out parity doubling. As an example of this procedure we have displayed a well-known model whose energy-momentum tensor may be written in terms of currents and is invariant under the chiral $SU(2)\times SU(2)$ group. The identity of $\theta_{(+)}{}^{\mu\nu}$ and $\theta_{(-)}{}^{\mu\nu}$ allows one to avoid the appearance of two independent Poincaré groups, and the ensuing parity doubling of the particle spectrum. In this regard, Eq. (7) is the crucial relation, and it would be interesting to see if it is possible to find other models in which it is satisfied. Finally, we note that there is no intrinsic difficulty in extending the model given to $SU(3)\times SU(3)$ or more complex groups. The essential ingredients as, given by Eqs. (14) and (15), are easily extended to these more general cases.

Note added in manuscript. After this work was complete we received an Imperial College Report (unpublished) by K. Barnes and C. Isham which also notes that nonlinear boson models can be used to construct a theory of currents with q-number Schwinger terms.

PHYSICAL REVIEW VOLUME 184, NUMBER 5 25 AUGUST 1969

Low-Energy Theorems on Radiative Corrections*

T. P. Cheng†
The Rockefeller University, New York, New York 10021
(Received 18 March 1969)

A general method is presented for evaluating, in a model-independent way, the soft-virtual-photon radiative corrections to an arbitrary hadronic process. It is shown that all the results concerning infrared divergences obtained within the theory of quantum electrodynamics of the electron-photon system are, in fact, exact in strong interactions. The problem of radiative corrections to low-energy theorems is the primary concern of this investigation. The threshold contributions of intermediate soft-photon states are nonanalytic in the photon frequency ω. While the procedure is general enough to permit, in principle, the calculation of the leading terms (as $\omega\to 0$) of these radiative corrections to all orders in e, in this paper only the leading e^2 radiative corrections are computed explicitly: They are of the order $\ln\omega$ for the bremsstrahlung (as first noted by Soloviev) and of the order $\omega\ln\omega$ for pion photoproduction. Accordingly, in the presence of radiative corrections, there are no longer any low-energy theorems for the $O(\omega^0)$ bremsstrahlung and $O(\omega)$ pion photoproduction amplitudes. The e^4 Compton amplitude $O(\omega^2\ln\omega)$ is computed and shown to be independent of the target spin.

I. INTRODUCTION

THE primary purpose of this paper is to discuss a number of low-energy theorems for amplitudes of Compton scattering, bremsstrahlung, and pion photoproduction. It is well known that, to the lowest order in the electric charge (e), low-energy theorems have been proved for these scattering processes. Here we examine the validity of these theorems in the presence of higher-order radiative corrections. In the low-frequency (ω) limit, the leading terms in the radiative corrections are shown to be structure-independent, i.e., there also exist low-energy theorems for radiative corrections themselves, valid to all orders in strong interactions.

As we shall see, all the radiative corrections to be discussed in this paper come from contributions of soft virtual photons. Our results are obtained by straightforward applications of the usual (i.e., lowest-order in e) low-energy theorems for processes involving virtual photons, and their validity rests exclusively on the general assumptions of Lorentz covariance and gauge invariance, as in the case of the usual theorems.

In the next section we will discuss the general problem of soft-photon radiative corrections and, in particular, their relevance to the question of the validity of low-energy theorems to higher orders in e. Our model-independent method for evaluating the leading terms of such radiative corrections will be outlined and the principal results stated. In the subsequent sections, details of the derivations will be given. In Sec. III, we discuss the problem of infrared-divergent radiative corrections to an arbitrary process involving charged hadrons. Radiative corrections to low-energy theorems will be calculated for bremsstrahlung in Sec. IV A, for pion photoproduction in Sec. IV B, and for Compton scattering in Sec. V.

II. OUTLINE OF THE PROBLEM

A. Low-Energy Theorems and Soft-Photon Problem

Let $\epsilon_\lambda T_\lambda$ be the invariant scattering amplitude of the process $\alpha\to\beta+\gamma$, with α and β being arbitrary hadron states and γ being a photon of momentum k and polarization ϵ_λ.[1] Gauge invariance requires that

$$k_\lambda T_\lambda = 0. \tag{2.1}$$

* Based on parts of a Ph.D. thesis to be submitted to the Rockefeller University.

† Present address: The Institute for Advanced Study, Princeton, N. J.

[1] The following argument can be generalized in an obvious way to cases where more than one low-frequency photon is involved.

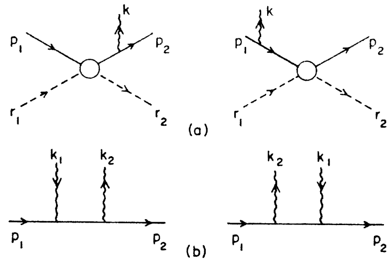


FIG. 1. Pole terms for (a) bremsstrahlung from scattering of a charged particle p with a neutral particle r and (b) Compton scattering.

Differentiating on both sides, we obtain

$$T_\lambda = -k_\mu[\partial T_\mu/\partial k_\lambda]. \tag{2.2}$$

The part of the amplitude for which the limit $[\partial T_\mu/\partial k_\lambda]_{k\to 0}$ exists (i.e., the part which is analytic at $k=0$) must be of the order k. Accordingly, gauge invariance implies that

$$T_\lambda = S_\lambda + O(k), \tag{2.3}$$

where S_λ is the part of T_λ which is singular at $k=0$. The separation of S_λ from T_λ can be made in a unique and gauge-invariant way, by adding whatever nonsingular terms are necessary to the singular part so that S_λ itself satisfies Eq. (2.1). Consequently, to obtain the low-energy behavior of the amplitude T_λ, our principal task will be to compute the gauge-invariant singular contributions.

To the lowest order in electromagnetism, the only nonanalytic (at $k=0$) term in the amplitude will be the single-particle pole term, which corresponds to the photon(s) being radiated from external charged-particle lines. Two simple examples of such pole terms are shown in Fig. 1.[2] These pole terms can be computed with the on-shell amplitudes of the original process without photon(s), giving the usual low-energy theorems.

On the other hand, in the presence of higher-order radiative corrections, the task of computing S_λ becomes more complicated, since contributions from intermediate states of particle plus zero-mass photons give rise to branch cuts extending down to $k=0$. To obtain the low-energy behavior of the amplitude T_λ, our compute not only the pole terms but also the threshold contributions of these intermediate soft-photon states. It will be shown, as one of our main results, that these nonanalytic threshold terms can again be calculated, as in the case of the pole terms, from the on-shell amplitude for scattering without radiation, i.e., the entire S_λ can be calculated in a structure-independent way. In this paper we will compute explicitly only the e^2 radiative correction, but the procedure is general enough to allow us, in principle, to calculate the higher-order threshold contributions.

[2] All the "graphs" we will draw in this paper have complete vertices; i.e., strong interactions have been included to all orders, so that we have various form factors.

It is clear that the validity of the usual low-energy theorems to higher orders in e will depend on what orders of photon frequencies these nonanalytic threshold terms are. For example, the existence of a $\omega\ln\omega$ term will then limit the original series expansion in powers of ω to no higher than ω^0, and so on. At the end of Sec. IV A, a more detailed discussion will be given of the question "What does one mean by 'validity' of the low-energy theorems to higher orders in e?"

Of course, infrared divergences are associated with the virtual soft-photon radiative corrections. It is clear that here two expansions are made of the scattering amplitudes: one in the electric charge e and another in the photon frequency ω. Soft photons complicate both these expansions: For the expansion in e we have the infrared divergences; for the expansion ω we have the nonanalytic terms—for example, $\omega\ln\omega$. Thus, the zero-mass property of the photon leads to infinities in the coefficients of both power-series expansions. Both these types of complications will appear in the discussion of higher-order radiative corrections to low-energy theorems.

In the next subsection we will give an outline of the general procedure for evaluating the radiative corrections due to soft virtual photons in an arbitrary hadronic process.

B. Method of Evaluation

The e^2 radiative correction amplitude $T(2)$ for the hadronic process $\alpha\to\beta$ can be represented as

$$T(2)=\frac{1}{2!}\int\frac{d^4k'}{(2\pi)^4}iD_{\mu\lambda}(k'^2)M_{\mu\lambda}. \tag{2.4}$$

$D_{\mu\lambda}(k'^2)$ is the propagator for the virtual photon with momentum k' and polarization indices λ and μ:

$$iD_{\mu\lambda}(k'^2)=\frac{\sum_{\epsilon'}\epsilon_\mu'\epsilon_\lambda'}{i(k'^2-i\epsilon)}. \tag{2.5}$$

$M_{\mu\lambda}$ is related to the matrix element of the electromagnetic current operator J by

$$M_{\mu\lambda}=-i\int d^4x\, e^{-ik'x}\,{}_{\rm out}\langle\beta|T(J_\mu(x)J_\lambda(0))|\alpha\rangle_{\rm in}+\rho_{\mu\lambda}, \tag{2.6}$$

where $\rho_{\mu\lambda}$ stands for possible "seagull" term which compensates the noncovariant nature of the T product and ensures that $M_{\mu\lambda}$ is gauge-invariant. $M_{\mu\lambda}$ is the amplitude for the process $\alpha+\gamma'\to\beta+\gamma'$, γ' being virtual photons.

Since we are only interested in the part of d^4k' integration corresponding to the four-vector $k'\to 0$, a number of simplifications can be made in the actual calculation. Since these simplifications will be used throughout this paper, each of them will now be discussed in some detail.

(1) Regardless of whether the photons are on- or off-shell, the low-energy theorem (to the lowest order in e) holds,[3] and it informs us (as discussed in Sec. II A) that the leading terms in $M_{\mu\lambda}$ for $k'\to 0$ come from diagrams in which the two virtual-photon lines are attached to the external charged particles. Figure 2 depicts such double-pole diagrams. For the special cases (which we will consider in Secs. IV and V) of soft-photon radiative correction to processes which themselves have soft photon(s) in their initial and final states, the relevant $M_{\mu\lambda}$ is then given by diagrams where all the soft photons, real and virtual, are emitted or absorbed from the external charged lines. Figure 3 is such an example.

It is principally due to this simplification that we are able to compute in a model-independent way the soft-photon radiative corrections to an arbitrary hadronic process.

(2) In this paper we assume that electromagnetic vertices exist for particles of arbitrary spin.[4] In the limit of $k_\lambda'\to 0$ the leading term for any vertex is the same:

$$\langle p_2|J_\lambda|p_1\rangle = ie(p_1+p_2)_\lambda\,,$$

with $k'=p_2-p_1$. All the spin-dependent[5] and off-shell[6] terms will be of higher orders in photon energy. Thus, if we are only interested in the leading term in soft-virtual-photon radiative corrections, we can simply work with spin-zero particles.

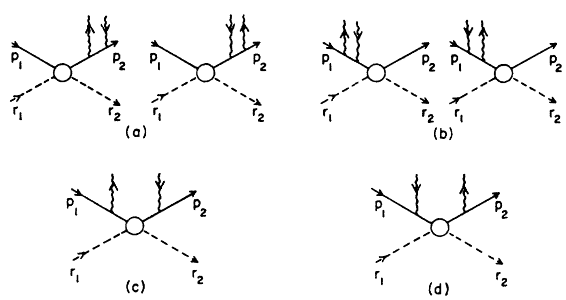
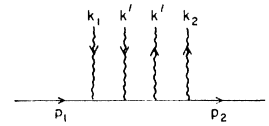


FIG. 2. Double-pole terms in scattering involving two low-frequency photons.

FIG. 3. External-line emission diagram for the process $p_1+k_1+k'\to p_2+k_2+k'$.



(3) $k_\lambda'\to 0$ implies $k'^2\to 0$; at the threshold the virtual photon γ' is really on its mass shell. In the $\int d\omega'$ integration only the contribution from the $\omega' = |\mathbf{k}'|$ pole survives (to the order we are interested in).[7] Accordingly, for the purpose of computing the radiative correction due to a *soft* virtual photon, the following substitution may be formally made:

$$\int \frac{d^4k'}{(2\pi)^4}\frac{1}{i(k'^2-i\epsilon)} \xrightarrow[k_\lambda'\to 0]{} \int \frac{d\mathbf{k}'}{(2\pi)^3}\frac{1}{2\omega'}\,, \qquad (2.7)$$

with $\omega'=|\mathbf{k}'|$. As we shall see, this replacement simplifies considerably the actual computation in subsequent sections.

C. Principal Results

Using the general procedure outlined above, we shall demonstrate in Sec. III that the results obtained with respect to infrared divergence within the theory of quantum electrodynamics are not modified when strong interactions are included to all orders.[8]

In Sec. IV the soft-photon radiative correction will be evaluated when the hadronic process also involves a low-frequency photon, $\alpha\to\beta+\gamma$. Besides giving the expected infrared-divergent factors, the soft virtual photons also bring about nonanalytic terms in the ω expansion: In Sec. IV A, for the bremsstrahlung ($\alpha\to\beta$ being some physically allowed process) we have a term proportional to $\ln\omega$, while the ω^0 term, for which Low[9] has proved the well-known low-energy theorem to order e, is shown to be structure-dependent in the presence of radiative corrections. The low-energy theorem for the $e^3\ln\omega$ bremsstrahlung amplitude obtained here agrees with the results derived by Soloviev.[10] In Sec. IV B, for pion photoproduction ($\alpha\to\beta$ being the pion-nucleon vertex) we have a term proportional to $\omega\ln\omega$. It is interesting to note that our result shows that the radiative correction to the ω term, for which Fubini *et al.*[11] have proved a PCAC (partially conserved axial-vector current) low-energy theorem, is structure-dependent.

In Sec. V our method is extended to include Compton scattering. The leading threshold contribution $O(e^4\omega^2\ln\omega)$ is shown to be completely determined by the target particle. This result has been previously

[3] The gauge conditions (2.1) and (2.2) remain true when $k^2\neq 0$.

[4] For some relevant comments in this connection see Ref. 4 of A. Pais, Phys. Rev. Letters **19**, 544 (1967).

[5] While this statement is clearly true for spin $\frac{1}{2}$, it is not so transparent for higher spins. For example, the spin-dependent term $\mathbf{S}(\mathbf{S},\mathbf{p}_1+\mathbf{p}_2)+(\mathbf{S},\mathbf{p}_1+\mathbf{p}_2)\mathbf{S}$ ($\mathbf{S}$ being the spin operator) in $\langle p_2|\mathbf{J}|p_1\rangle$ cannot be ruled out simply by rotational invariance. However, L. I. Lapidus and Chou Kuang-Chao, Zh. Eksperim. i Teor. Fiz. **39**, 1286 (1960) [English transl.: Soviet Phys.—JETP **12**, 898 (1961)], have shown that such factors are absent in the low-frequency matrix elements of the current by requiring that J_λ transform as a Lorentz four-vector. For a systematic discussion of cases of high spin, see A. Pais, Nuovo Cimento **53**, 433 (1968). We will parametrize the electromagnetic vertices of spinning particles in such a way that their spin-independent nature in the $k'\to 0$ limit is clearly displayed, e.g., for spin $\frac{1}{2}$, we write

$$\langle p_2|J_\lambda|p_1\rangle = \bar{u}(p_2)[f_1(k'^2)i(p_1+p_2)_\lambda + f_2(k'^2)\sigma_{\lambda\mu}k_\mu']u(p_1).$$

[6] Since we have soft photons (real or virtual) attached to external charged particles only, the off-shell terms must necessarily vanish in the zero-frequency limit.

[7] For an evaluation of similar types of integrals, see, for example, J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill Book Co., New York, 1964), p. 173.

[8] Similar results have been proved independently in renormalizable theories by S. R. Choudhury (private communication).

[9] F. E. Low, Phys. Rev. **110**, 974 (1958).

[10] L. D. Soloviev, Nucl. Phys. **64**, 657 (1965).

[11] S. Fubini, G. Furlan, and C. Rossetti, Nuovo Cimento **40**, 1171 (1965).

derived via dispersion relations for the forward differential cross section by Gerasimov and Soloviev,[12] then for the amplitude by Roy and Singh,[13] and, independently, by the present author.[14] It is rederived here without extraneous high-energy assumptions. The simplicity of the actual computation allows us to see immediately that the results for the spin-zero case actually hold for targets of arbitrary spin.[15]

III. SOFT-PHOTON RADIATIVE CORRECTIONS TO $\alpha \rightarrow \beta$: INFRARED DIVERGENCES

The physical basis of infrared divergences was elucidated long ago starting with the work of Bloch and Nordsieck.[16] They treated a simplified model in which a fixed classical current density interacts with a quantized electromagnetic field. The virtue of this model is that it can be solved (without recourse to the usual perturbation method), and thus leads to insight into the difficulties brought about by photons in the long-wavelength region. The principal conclusion is the following: When a charged particle is scattered, it always radiates an infinite number of soft photons (with finite total energy). The cross section of any process involving charged particles and a definite number of photons, for which the perturbation expansion in the fine-structure constant is not valid, is an unphysical quantity. What is measured in reality is the probability for scattering with an energy loss (due to radiation) of less than ΔE, which is the energy resolution of the experimental setup. A perturbation expansion for this physically observable quantity is, however, permissible; the infrared divergence in the radiative correction and the corresponding divergence in the cross section for photon emissions cancel order by order.

This cancellation of the infrared divergence in the cross sections has been demonstrated explicitly for quantum electrodynamics of the electron-photon system. The proof, as given, for instance, by Jauch and Rohrlich,[17] depends on some special properties of quantum electrodynamics. Consequently, we must inquire whether the cancellation remains true for the case of hadrons if strong interactions are included to all orders. We will show that this is indeed the case, as is to be anticipated.

Consider the process $\alpha \rightarrow \beta$, where α and β are some arbitrary hadron states involving charged particles. It is well known that the infrared-divergent part of the cross section $d\sigma$ for $\alpha \rightarrow \beta+\gamma$ can be factored out and is proportional to the $d\sigma$ for $\alpha \rightarrow \beta$. This is just the usual ω^{-1} bremsstrahlung low-frequency theorem. Our main interest is therefore the factorization of infrared-divergent radiative corrections to $\alpha \rightarrow \beta$ due to virtual photons.

For definiteness, consider the case where $\alpha(\beta)$ consists of one charged particle p_1 (p_2) and one neutral particle r_1 (r_2).[18] According to the general method of evaluating the soft-photon radiative corrections as outlined in Sec. II, we first consider the process $p_1+r_1+\gamma' \rightarrow p_2+r_2+\gamma'$ with the virtual photons γ' attached to the external charged lines p_1 and p_2 as shown in Fig. 2. We treat the two photons as distinguishable, and the overcounting will be compensated by the appropriate combinatorial factors. For graphs in Figs. 2(a) and 2(b) we shall, for the moment, imagine that the two photons have different momenta, k_1' and k_2', so that the contributions from these two graphs will be unambiguous.[19,20] For $k_1' \rightarrow 0$ and $k_2' \rightarrow 0$, we have from Fig. 2(a)

$$a_{\mu\lambda} = +e^2 \frac{p_{2\mu} p_{2\lambda}}{p_2 \cdot (k_2'-k_1')} \left(\frac{1}{p_2 \cdot k_2'} - \frac{1}{p_2 \cdot k_1'} \right) T(0) , \quad (3.1)$$

where $T(0)$ is the amplitude (to the lowest order in e) for $p_1+r_1 \rightarrow p_2+r_2$. Setting $k'=k_1'=k_2'$, we have

$$a_{\mu\lambda} = -e^2 \frac{p_{2\mu} p_{2\lambda}}{(p_2 \cdot k')(p_2 \cdot k')} T(0) . \quad (3.2)$$

For Fig. 2(c) we obtain

$$c_{\mu\lambda} = +e^2 \frac{p_{2\mu} p_{1\lambda}}{(p_2 \cdot k')(p_1 \cdot k')} T(0) . \quad (3.3)$$

[12] S. B. Gerasimov and L. D. Soloviev, Nucl. Phys. **74**, 589 (1965).

[13] S. M. Roy and V. Singh, Phys. Rev. Letters **21**, 861 (1968).

[14] T. P. Cheng, Phys. Rev. **176**, 1674 (1968).

[15] This result is known independently to K. Y. Lin (private communication).

[16] F. Bloch and A. Nordsieck, Phys. Rev. **52**, 54 (1937).

[17] J. M. Jauch and F. Rohrlich, Helv. Phys. Acta **27**, 613 (1954); *The Theory of Photons and Electrons* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1955), Chap. 16.

[18] The particles are allowed to have arbitrary spin, the results being spin-independent.

[19] This is just a convenient device to avoid involvement in any discussion of the renormalization problem which would only obscure the simple kinematical properties being presented here (see Ref. 20). The contribution of Fig. 2(a) as given in Eqs. (3.1) and (3.2) is identical to the result obtained by an appropriate differentiation of (the integrand of) the electromagnetic self-energy diagram $\Sigma(p)$. After the virtual-photon integration, it gives the infrared-divergent factors of the wave-function renormalization constant Z_2.

[20] In quantum electrodynamics of the electron-photon system, infrared divergences are usually discussed together with the renormalization problems of the theory, since some of the renormalization constants are themselves infrared-divergent. Thus graphs where the virtual-photon line is not attached at both ends to external charged particles (hence noninfrared according to the criteria as stated above) often become infrared-divergent after renormalization. Accordingly, the following question has been raised by Dr. S. R. Choudhury in a private discussion: For a general scattering process, does renormalization introduce *extra* infrared-divergent terms in the physical amplitudes—that is, above and beyond those coming from graphs where the virtual-photon line is attached at both ends to external charged particles? We note that this problem arises from the fact that the renormalization constants are usually defined at points involving particles on their mass shell. If we had chosen different points (as we have the freedom to do) where all the charged particles in question are off-shell, no extra infrared divergences would be produced to begin with, thus eliminating the issue of the introduction and cancellation of such spurious divergences. In other words, renormalization can only shuffle infrared divergences from one part of the amplitude to another. No new factors are introduced in the full amplitude. Consequently, as far as soft-photon radiative corrections are concerned, we can consistently ignore the problem of renormalization and still get the correct results.

Substituting $p_1 \leftrightarrow p_2$ in $a_{\mu\lambda}$ and $c_{\mu\lambda}$ results in $b_{\mu\lambda}$ and $d_{\mu\lambda}$. The whole amplitude is then

$$M_{\mu\lambda} = -e^2\left(\frac{p_{2\mu}}{p_2\cdot k'}-\frac{p_{1\mu}}{p_1\cdot k'}\right)\left(\frac{p_{2\lambda}}{p_2\cdot k'}-\frac{p_{1\lambda}}{p_1\cdot k'}\right)T(0)\,. \quad (3.4)$$

We have for the e^2 radiative correction[21]

$$T(2) = -\tfrac{1}{2}e^2\int \frac{d\mathbf{k}'}{(2\pi)^3}\frac{1}{2\omega'}\left(\frac{p_2}{p_2\cdot k'}-\frac{p_1}{p_1\cdot k'}\right)^2 T(0)\,. \quad (3.5)$$

The interference term with $T(0)$ gives the e^2 cross section, which just cancels the corresponding cross section for the bremsstrahlung of a soft photon, $r_1+p_1 \to r_2+p_2+\gamma$. It is clear that the above result can easily be extended to include cases of $\alpha\to\beta$ with more complicated charge configurations.

For later reference we note that the infrared-divergent factor in Eq. (3.5) can be computed to be

$$T(2) = b(t)\int \frac{d\omega'}{\omega'}T(0)\,, \quad (3.6)$$

where $t = -(p_2-p_1)^2$. For the equal-mass case ($p_1{}^2 = p_2{}^2$), b takes on a simple form:

$$b(t) = \frac{\alpha}{\pi}\left(1+\frac{t-2m^2}{D^{1/2}}\ln\left|\frac{t-D^{1/2}}{t+D^{1/2}}\right|\right), \quad (3.7)$$

where $D = t^2-4tm^2$, and for $t\to 0$ we have

$$b(t) = \frac{\alpha}{3\pi}\frac{t}{m^2}+O(t^2)\,. \quad (3.8)$$

It should be noted that the validity of our result as stated in Eqs. (3.5) or (3.6) rests on the generality of our procedure in applying the low-energy theorem to the amplitude $\alpha+\gamma'\to\beta+\gamma'$ in the integrand. The results should be true in any local-field theory, and we note that a minimal effective electromagnetic vertex is not required here.[22]

IV. SOFT-PHOTON RADIATIVE CORRECTIONS TO $\alpha\to\beta+\gamma$

We now consider the slightly more complicated problem of soft-virtual-photon radiative corrections to the process $\alpha\to\beta+\gamma$, the emitted photon also being soft ($k_\lambda\to 0$). Here we expect to obtain not only the infrared-divergent terms but also the nonanalytic threshold factors ($\sim\ln\omega$) of the intermediate soft-photon state. Similarly, as in Sec. III, we first consider the process of $\alpha+\gamma'\to\beta+\gamma+\gamma'$. All the photons being soft, the low-energy theorem informs us that the leading contribution comes from those graphs in which all *three* photon lines, real and virtual, are attached to the external charged particles in the basic process $\alpha\to\beta$.

[21] We use Feynman gauge $\sum_{\epsilon'}\epsilon_\mu'\epsilon_\lambda' = \delta_{\lambda\mu}$.

[22] Z. Z. Aydin and A. O. Barut [Trieste Report No. IC/68/101 (unpublished)] have stated that the quantum-electrodynamic results with respect to infrared divergences will no longer hold when nonminimal Pauli coupling is used. However, these authors did not consider the self-energy diagrams in their calculation of the radiative correction to a proton electromagnetic vertex.

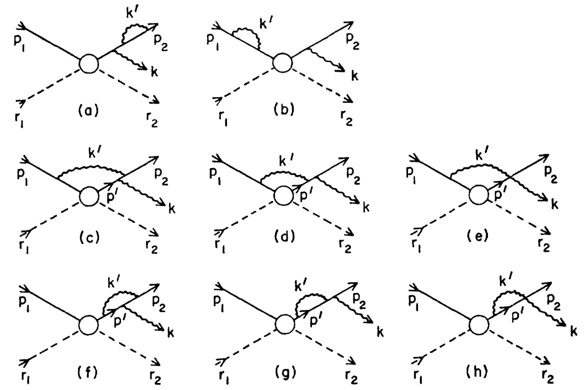


FIG. 4. Soft-photon radiative correction to bremsstrahlung. There are eight more crossed diagrams corresponding to the photon k being emitted from the initial charged line. They are related to the above diagrams by the substitutions $p_1\leftrightarrow p_2$ and $k_\lambda\leftrightarrow -k_\lambda$.

A. Bremsstrahlung Low-Energy Theorems

We will again consider, as in Sec. III, the case where the α (β) state consists of one charged particle p_1 (p_2) and one neutral particle r_1 (r_2). By the above argument it is then sufficient for our calculation of the soft-virtual-photon radiative corrections to consider the set of diagrams in Fig. 4. We have included seagull diagrams so that the total contribution in the soft-photon limit will be gauge-invariant.[23]

It is clear that Figs. 4(a) and 4(b) contribute only to the infrared-divergent terms:

$$(a+b)_\lambda = -\frac{e^3}{(2\pi)^3}\int\frac{d\mathbf{k}'}{2\omega'}\left(\frac{p_{2\mu}p_{2\mu}}{2(p_2\cdot k')^2}+\frac{p_{1\mu}p_{1\mu}}{2(p_1\cdot k')^2}\right)\times\left(\frac{p_{2\lambda}}{p_2\cdot k}T(0)\right), \quad (4.1)$$

where $T(0)$ denotes the zeroth-order-in-e, on-shell amplitude for the process $r_1+p_1\to r_2+p_2$ without photon emission.

The contribution from Figs. 4(c)–4(h) can be written in simple form:

$$(c+d+\cdots h)_\lambda = -\frac{e^3}{(2\pi)^3}\int\frac{d\mathbf{k}'}{2\omega'}\frac{1}{p'^2+m^2}\times 2\left(\frac{p_{2\mu}p_\lambda'}{p_2\cdot k'}-\frac{p_{2\lambda}p_\mu'}{p_2\cdot k}+\delta_{\lambda\mu}\right)\left(\frac{p_\mu'}{p_2\cdot k}-\frac{p_{1\mu}}{p_1\cdot k'}\right)T(0)\,, \quad (4.2)$$

[23] Because of the manner with which we parametrize the vertsx (see Ref. 5), there are seagull-type diagrams even for spin-$\frac{1}{2}$ particles. They represent simply the nonsingular contributioen (from the continuum) that are added to the singular term to form a gauge-invariant (low-frequency) amplitude (see discussion in Sec. II A).

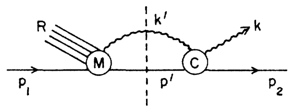


FIG. 5. Singular contributions by the state of a particle plus a soft photon. The vertical dashed line indicates that k' and p' are on-shell. R stands for an arbitrary number of neutral particles.

where $p'=p_2+k-k'$. The $\ln\omega$ terms come about because the propagator $(p'^2+m^2)^{-1}$ diverges as $k\to 0$ and $k'\to 0$. Before proceeding with the computation of these terms, we will first separate out the infrared-divergent factor coming from Fig. 4(c) which is proportional to $(p_2\cdot k')^{-1}(p_1\cdot k')^{-1}$. This can be accomplished easily by noting that

$$\frac{1}{p'^2+m^2}\times\frac{1}{p_2\cdot k'}=\left(\frac{1}{p'^2+m^2}+\frac{1}{2p_2\cdot k'}\right)\frac{1}{p_2\cdot k-k'k}\,.$$

Dropping terms that contribute neither to the divergent factor not to the $\ln\omega$ terms, we have

$$\begin{aligned}(c+d+\cdots h)_\lambda
&=-\frac{1}{(2\pi)^3}\int\frac{d\mathbf{k}'}{2\omega'}\Bigg[\left(-e^2\frac{p_{1\mu}p_{2\mu}}{(p_1\cdot k')(p_2\cdot k')}\right)\frac{p_{2\lambda}}{p_2\cdot k}eT(0)\\
&\quad+\frac{2e^2}{p'^2+m^2}\left(\frac{p_{2\mu}p_\lambda'}{p'\cdot k}-\frac{p_{2\lambda}p_\mu'}{p_2\cdot k}+\delta_{\lambda\mu}\right)\\
&\qquad\times\left(\frac{p_\mu'}{p_2\cdot k}-\frac{p_{1\mu}}{p_1\cdot k'}\right)eT(0)\Bigg]\,. \qquad (4.3)\end{aligned}$$

The first term combined with $(a+b)_\lambda$ in Eq. (4.1) gives the expected infrared-divergent factors, which are proportional to the lowest-order bremsstrahlung amplitude.

In the following discussion we will concentrate on the $\ln\omega$ factors coming from the second term in Eq. (4.3). Since these logarithms are brought about by the integration over the $(p'^2+m^2)^{-1}$ pole terms, we are allowed to set $p'^2+m^2=0$ in the numerator, which is the residue of the pole. Accordingly, the nonanalytic terms can be written in a general form:

$$-\frac{1}{(2\pi)^3}\int\frac{d\mathbf{k}'}{2\omega'}\Big(\sum_{\epsilon'}\epsilon_\mu'\epsilon_\nu'\Big)\times\frac{C_{\lambda\mu}(k,p_2;k',p')M_\nu(k',p';p_1)}{p'^2+m^2}\,, \qquad (4.4)$$

where M_ν is the on-shell amplitude for the bremsstrahlung itself and $C_{\lambda\mu}$ is the on-shell Compton amplitude. Both amplitudes are of the lowest order in e. The variables in parentheses specify the relevant momenta for these processes.

It should be noted in passing that, in general, M_ν represents the amplitude for any soft-photon process for which the radiative correction is in question. A diagrammatic representation of this statement is given in Fig. 5. In the Appendix, Eq. (4.4) will be derived using the noncovariant Low equation, so as to display the direct relationship of our calculation with Low's method.

Returning to the calculation at hand, keeping only the leading terms in the integrand, i.e., $O(\omega^0)$ for $C_{\lambda\mu}$ and $O(\omega^{-1})$ for M_ν. We have from Eq. (4.3)

$$\begin{aligned}(c'+d+\cdots h)_\lambda
&=\frac{e^3T(0)}{(2\pi)^3}\int\frac{d\mathbf{k}'}{2\omega'}\frac{1}{p_2\cdot(k-k')}\Bigg[\left(k_\lambda'-p_{2\lambda}\frac{k'\cdot k}{p_2\cdot k}\right)\left(\frac{p_2}{p_2\cdot k}\right)^2\\
&\quad+\frac{1}{p_1\cdot k'}\left(p_{1\lambda}-p_{2\lambda}\frac{p_1\cdot k}{p_2\cdot k}\right)\\
&\quad-\frac{1}{p_1\cdot k'}\left(k_\lambda'-p_{2\lambda}\frac{k'\cdot k}{p_2\cdot k}\right)\frac{p_1\cdot p_2}{p_2\cdot k}\Bigg]\,. \qquad (4.5)\end{aligned}$$

The prime over the amplitude c_λ indicates that the infrared-divergent factor has been extracted. The first term vanishes after the $d\mathbf{k}'$ integration, since

$$\int\frac{d\mathbf{k}'}{2\omega'}\frac{k_\lambda'}{p_2\cdot(k-k')}$$

must be proportional to $p_{2\lambda}$; therefore we have only two types of integrals to calculate:

$$\int\frac{d\mathbf{k}'}{2\omega'}\frac{1}{p_2\cdot(k-k')}\frac{1}{p_1\cdot k'}\equiv A\ln|p_2\cdot k| \qquad (4.6)$$

and

$$\int\frac{d\mathbf{k}'}{2\omega'}\frac{1}{p_2\cdot(k-k')}\frac{k_\lambda'}{p_1\cdot k'}\equiv[Bp_{1\lambda}+Cp_{2\lambda}]\ln|p_2k|\,, \qquad (4.7)$$

giving

$$(c'+d+\cdots h)_\lambda=\frac{e^3T(0)}{(2\pi)^3}\left(p_{1\lambda}-p_{2\lambda}\frac{p_1\cdot k}{p_2\cdot k}\right)\times\left(A-B\frac{p_1\cdot p_2}{p_2\cdot k}\right)\ln|p_2\cdot k|\,. \qquad (4.8)$$

A and B can be evaluated easily by going first to the rest frame of p_2:

$$A=-4\pi D^{-1/2}\ln\left|\frac{t-D^{1/2}}{t+D^{1/2}}\right|\,, \qquad (4.9)$$

$$B=4\pi(p_2\cdot k)D^{-1}\left(2-4p_1\cdot p_2D^{-1/2}\ln\left|\frac{t-D^{1/2}}{t+D^{1/2}}\right|\right)\,, \qquad (4.10)$$

where t and D are defined as in Sec. III. It is then straightforward to check that

$$\frac{e^2}{(2\pi)^3}\left(A-B\frac{p_1\cdot p_2}{p_2\cdot k}\right)=-2b'(t)\,, \tag{4.11}$$

b' being the derivative with respect to the variable t of the function b defined in Eq. (3.6). Adding the contribution from the crossed term and noting the relation $\ln|p_1\cdot k|=\ln|p_2\cdot k|+O(k^0)$, we obtain from the soft-photon radiative correction to the bremsstrahlung amplitude

$$M_\lambda(3)=e\left(\frac{p_{2\lambda}}{p_2\cdot k}-\frac{p_{1\lambda}}{p_1\cdot k}\right)T(0)\left(b(t)\int\frac{d\omega'}{\omega'}\right.$$
$$\left.-2b'(t)(p_2-p_1)\cdot k\ln|p_1\cdot k|\right)+O(k^0)\,. \tag{4.12a}$$

When the velocities of the charged particles are nonrelativistic and $p_1^2=p_2^2=-m^2$, $t\approx-(\mathbf{p}_2-\mathbf{p}_1)^2\equiv-\mathbf{\Delta}^2$. Hence the angular factor $b'(t)$ can be expanded and Eq. (4.12) is reduced to the following simple form:

$$\mathbf{M}(3)=\frac{-e^3}{12\pi^2\omega m^3}\mathbf{\Delta}\left(\mathbf{\Delta}^2\int\frac{d\omega'}{\omega'}-2\mathbf{\Delta}\cdot\mathbf{k}\ln\omega\right)T(0)\,. \tag{4.12b}$$

This is the result first obtained by Soloviev.[10] Our argument (2) in Sec. II B makes it clear that Eq. (4.12) also holds for cases of higher spin. It agrees with the low energy limit of the exact relativistic calculation in perturbation theory by Fomin[24] for the process of bremsstrahlung by an electron scattered in Coulomb field.

With this concrete expression for the low-energy behavior of the e^3 bremsstrahlung amplitude, we can now give a more detailed discussion of the question "What does one mean by the 'validity' of the low-energy theorem to higher orders in electromagnetism?"

The theorem proved by Low[9] states that not only the leading ω^{-1} term but also the next-order ω^0 term in the bremsstrahlung amplitude (M_λ) can be computed from the corresponding scattering amplitude without photon emission (T). For definiteness let us take the case of spin-zero scattering:

$$M_\lambda=e\left(\frac{p_{2\lambda}}{p_2\cdot k}-\frac{p_{1\lambda}}{p_1\cdot k}\right)T$$
$$+e\left[\frac{r_2\cdot k}{p_2\cdot k}p_{2\lambda}-r_{2\lambda}+\frac{r_1\cdot k}{p_1\cdot k}p_{1\lambda}-r_{1\lambda}\right]\frac{\partial T}{\partial\nu}\,, \tag{4.13}$$

where $\nu=-(p_1\cdot r_1+p_2\cdot r_2)$. The proof is carried out only to the lowest order in e: $M_\lambda=M_\lambda(1)$ and $T=T(0)$. (The number in parentheses denotes the order in e.) If the theorem were valid to the nth order in α, then the relation (4.13) would still hold, with M_λ and T taken to the corresponding orders: $M_\lambda=M_\lambda(1)+\cdots+M_\lambda(2n+1)$ and $T=T(0)+\cdots+T(2n)$. The radiative corrections to M_λ and T naturally include infrared-divergent factors. This is precisely what we have in Eq. (4.12): The pole term is multiplied by an infrared-divergent factor. The corresponding divergent factors on both sides are identical, as they necessarily must be. Thus we say that the infrared-divergent radiative corrections to the low-energy theorems are really of a trivial kind. They do not change the form of the low-energy theorems derived in the lowest order. To be sure, due to the existence of these infrared factors, ΔE will have to be introduced in the cross sections. But this is a general feature of any physical measurement and is not limited in any way to the low-energy behavior of scattering.

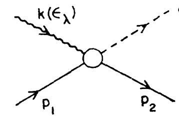


FIG. 6. Pion photoproduction.

We now see that it is the nonanalytic factor ($\sim\omega^n\ln\omega$) which is crucial to the validity of the higher-order theorems. It invalidates the original expansion in ω. A more concrete way of viewing the problem is to note that as in the case of bremsstrahlung the existence of threshold terms $\ln\omega$ indicates that the nonthreshold contributions are of the order ω^0, its precise value being cutoff-dependent. Accordingly, there is no longer any low-energy theorem for the ω^0 term in the e^3 amplitude, since soft photons introduce structure-dependent terms of this order.

B. Pion Photoproduction Low-Energy Theorems

For $\alpha\to\beta+\gamma$, with $\alpha\to\beta$ being a pion-nucleon vertex, we have the simple but physically interesting case of pion photoproduction. Figure 6 shows the kinematics. The pion has isospin index a and mass μ.

We are interested in the limit of $k\to0$ and $q\to0$, i.e., the low-energy limit in the no-recoil approximation. To the lowest order in e, Kroll and Ruderman[25] have shown that the leading ω^0 term in the amplitude is completely determined by the electric charge and the pion-nucleon coupling constant g. In the laboratory system, the theorem reads

$$\mathbf{M}^a=i\frac{eg}{4m}\boldsymbol{\sigma}[\tau^a,\tau^3]+O(\omega)+O(\mu)\,. \tag{4.14}$$

We can follow the procedure used in Sec. IV A to obtain the radiative corrections. To order e^3 the pole term is still the same as in Eq. (4.14), with g being the e^2 radiative correction to the pion-nucleon coupling constant, including the expected infrared-divergent factors. As for the nonanalytic threshold terms, we can simply make use of the general expression (4.4), with M_λ being the lowest-order pion photoproduction amplitude. Since the initial and final momenta of the

[24] P. I. Fomin, Zh. Eksperim. i Teor. Fiz. 35, 707 (1958) [English transl.: Soviet Phys.—JETP 35, 491 (1959)].

[25] N. M. Kroll and M. A. Ruderman, Phys. Rev. 93, 233 (1954).

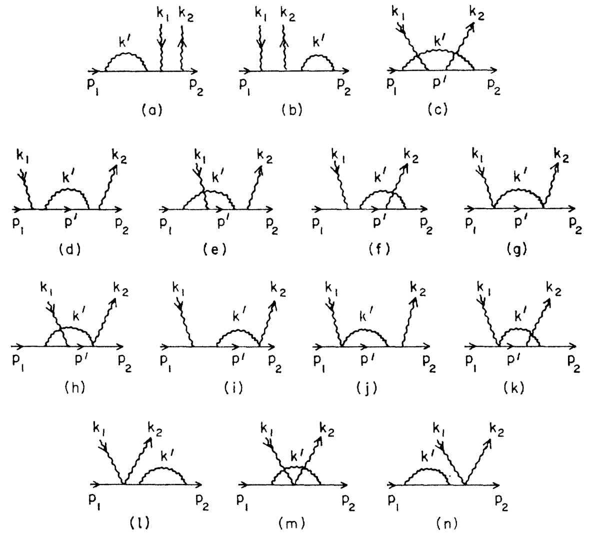


FIG. 7. Soft-photon radiative correction to Compton scattering. There are eleven more crossed diagrams corresponding to an exchange of k_1 and k_2 in diagrams (a)–(k). The contributions of the crossed diagrams can be obtained by the substitution $k_{1\lambda} \leftrightarrow k_{2\mu}$.

nucleon are constrained in this scattering, the calculation is considerably simplified when performed in the laboratory system and with the transverse photon gauge.[26] Evaluating the integrand by the Thomson theorem for the Compton amplitude and by Eq. (4.14), we have

$$\boldsymbol{\varepsilon}\cdot\mathbf{M}^a(3)=\frac{-1}{(2\pi)^3}\left(i\frac{e^3g}{2m^2}\right)\int_0\frac{\omega' d\omega'}{2}\int d\Omega'\sum_{\epsilon'}(\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}')(\boldsymbol{\varepsilon}'\cdot\boldsymbol{\varepsilon})\times\left(\frac{\frac{1}{4}[\tau^a,\tau^3](1+\tau^3)}{\omega'-\omega}+\frac{\frac{1}{4}(1+\tau^3)[\tau^a,\tau^3]}{\omega'+\omega}\right), \quad (4.15)$$

which yields the threshold contribution

$$\left(i\frac{eg}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}\right)\frac{2\alpha}{3\pi}\frac{\omega}{m}\ln\omega\,\tau^a(1-\delta_{3a})+O(\omega)+O(\mu). \quad (4.16)$$

The $\omega \ln\omega$ term comes in two isospin channels. The term proportional to τ^a is one of the amplitudes for which Fubini *et al.* have given a PCAC low-energy theorem to order ω. Our result shows that this term is structure-dependent when radiative corrections are included, for the same reason as was discussed in Sec. IV A for the ω^0 term in bremsstrahlung amplitudes. The independent isospin amplitude $\tau^a\delta_{3a}$ appears because the intermediate photon has an isovector part.

V. SOFT-PHOTON RADIATIVE CORRECTIONS TO $\alpha+\gamma\to\beta+\gamma$: COMPTON-SCATTERING LOW-ENERGY THEOREMS

Compton scattering is the simplest two-photon process $\alpha+\gamma\to\beta+\gamma$: α and β are merely single charged-particle states.

It has long been known that to the lowest order in e the leading ω^0 and ω Compton amplitudes are completely determined by the charge and magnetic moment of the target particle.[27] These Compton-scattering low-energy theorems are of particular importance. Besides giving us an empirical definition of the electromagnetic couplings, these theorems, in conjunction with certain high-energy assumptions, may be converted into a group of interesting sum rules. In this section the low-energy behavior of the e^4 Compton amplitude is investigated.

We begin the calculation by considering the amplitude for the process

$$p_1+k_1(\epsilon_{1\lambda})+k'(\epsilon_\nu')\to p_2+k_2(\epsilon_{2\mu})+k'(\epsilon_\rho')$$

(which also serves to define the kinematics of the problem). The target p is allowed to have arbitrary spin. As we shall see, the $O(e^4\omega^2\ln\omega)$ Compton low-energy theorem to be derived in this section is spin-independent. When all photons have small energy, the leading terms are associated with diagrams which can be obtained by making all possible permutations of the four photon lines in Fig. 3. We then tie the k' lines together, and integrate (with the integration range $0\leq|\mathbf{k}'|<\delta$, $\delta\to 0$) to obtain the soft-virtual-photon radiative correction to the low-energy Compton-scattering amplitude [see Fig. 7; again we have included the seagull graphs[23] in Figs. 7(g)–7(n)].

Just as for pion photoproduction, the actual computation is greatly simplified when it is performed in a special Lorentz frame ($\mathbf{p}_1=0$) with a particular gauge (the Coulomb gauge).

In the laboratory system, the initial and final photon frequencies are related by

$$\omega_1-\omega_2=(\omega_1\omega_2/m)(1-\cos\Theta), \quad (5.1)$$

Θ being the laboratory scattering angle. Whenever the difference $\omega_1-\omega_2$, which is quadratic in photon frequency, can be neglected, we will simply refer to both ω_1 and ω_2 as ω.

It is clear that Figs. 7(l)–7(n) and 7(a)–7(c), with their cross graphs, give rise to the infrared divergence. By Eqs. (3.6) and (3.8) [the momentum transfer $t=-2\omega_1\omega_2(1-\cos\Theta)$ vanishes in the soft-photon limit],

$$-\frac{2\alpha}{3\pi}\frac{\omega^2}{m^2}(1-\cos\Theta)\int\frac{d\omega'}{\omega'}C^{(2)}, \quad (5.2)$$

where $C^{(2)}$ is the lowest-order Compton amplitude.

We will now concentrate on the evaluation, in Figs. 7(c)–7(k), of the singular threshold contributions from intermediate soft-photon states. For this, we can

[26] Transverse gauge: $\sum_{\epsilon'}\epsilon_i'\epsilon_j'=\delta_{ij}-k_i'k_j'/|\mathbf{k}'|^2$ $(i, j=1, 2, 3)$.

[27] W. Thirring, Phil. Mag. **41**, 1193 (1950); F. E. Low, Phys. Rev. **96**, 1428 (1954); M. Gell-Mann and M. L. Goldberger, *ibid.* **96**, 1433 (1954). Recently, these theorems have been further extended, to scattering involving "charged" photons [M. A. B. Bég, Phys. Rev. Letters **17**, 333 (1966)], to parts of the $O(\omega^2)$ amplitude [V. Singh, *ibid.* **19**, 730 (1967)], and to cases of higher-spin targets (A. Pais, Refs. 4 and 5).

make direct use of Eq. (4.4):

$$C^{(4)\prime}=-\frac{\epsilon_{1\lambda}\epsilon_{2\mu}}{(2\pi)^3}\int\frac{d\mathbf{k}'}{2\omega'}\Big(\sum_{\epsilon'}\epsilon_\nu'\epsilon_\rho'\Big)\times\frac{C_{\mu\nu}^{(2)}(k_2,p_2;k',p')C_{\rho\lambda}^{(2)}(k',p';p_1,k_1)}{p'^2+m^2},\quad(5.3)$$

where $p'=p_1+k_1-k'$. The numerator is the product of two physical e^2 Compton amplitudes. To the order we are interested in, we have

$$\epsilon_\rho'C_{\rho\lambda}^{(2)}\epsilon_{1\lambda}=2e^2\boldsymbol{\varepsilon}_1\cdot\boldsymbol{\varepsilon}'+\omega S[\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}']+O(\omega^2)\quad(5.4)$$

and

$$\epsilon_{2\mu}C_{\mu\nu}^{(2)}\epsilon_\nu'=2e^2[\boldsymbol{\varepsilon}_2\cdot\boldsymbol{\varepsilon}'+(\mathbf{k}_1\cdot\boldsymbol{\varepsilon}')(\mathbf{k}_1\cdot\boldsymbol{\varepsilon}_2)/m\omega'-(\mathbf{k}_1-\mathbf{k}')\cdot\boldsymbol{\varepsilon}_2(\mathbf{k}_1-\mathbf{k}_2)\cdot\boldsymbol{\varepsilon}'/m\omega_2]+\omega S[\boldsymbol{\varepsilon}_2,\boldsymbol{\varepsilon}']+O(\omega^2),\quad(5.5)$$

where $\omega S[\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}']$ stands for possible spin-dependent terms. S by itself is of order ω^0 and is *odd* under crossing. As we shall see, none of these spin-dependent factors will contribute to the final results with respect to the radiative correction $O(\omega^2\ln\omega)$.

Thus, in the laboratory system, with transverse gauge,[26] Eq. (5.3) reads

$$C^{(4)\prime}=\frac{4e^4}{(2\pi)^3}\int\frac{d\mathbf{k}'}{2\omega'}\sum_{\epsilon'}\Big(\frac{1}{2m[\omega_1-\omega'-(\omega_1\omega'-\mathbf{k}_1\cdot\mathbf{k}')/m]}\times\{(\boldsymbol{\varepsilon}_1\cdot\boldsymbol{\varepsilon}')(\boldsymbol{\varepsilon}_2\cdot\boldsymbol{\varepsilon}')-(\boldsymbol{\varepsilon}_1\cdot\boldsymbol{\varepsilon}')(\mathbf{k}_1-\mathbf{k}')\cdot\boldsymbol{\varepsilon}_2(\mathbf{k}_1-\mathbf{k}_2)\cdot\boldsymbol{\varepsilon}'/m\omega_2+(\boldsymbol{\varepsilon}_1\cdot\boldsymbol{\varepsilon}')(\mathbf{k}_1\cdot\boldsymbol{\varepsilon}')(\mathbf{k}_1\cdot\boldsymbol{\varepsilon}_2)/m\omega'+\omega(\boldsymbol{\varepsilon}_1\cdot\boldsymbol{\varepsilon}')S[\boldsymbol{\varepsilon}_2,\boldsymbol{\varepsilon}']+\omega S[\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}'](\boldsymbol{\varepsilon}_2\cdot\boldsymbol{\varepsilon}')+O(\omega^2)\}\Big).\quad(5.6)$$

The only term we have to be careful with is the one proportional to $(\boldsymbol{\varepsilon}_1\cdot\boldsymbol{\varepsilon}')(\boldsymbol{\varepsilon}_2\cdot\boldsymbol{\varepsilon}')$; here we must keep the recoil terms proportional to $k_1\cdot k'$ in the denominator. Concentrating on the threshold contribution of this term, we have

$$\frac{e^4}{(2\pi)^3}\frac{1}{m}\sum_{\epsilon'}\int_0\frac{\omega'd\omega'd\Omega'}{\omega_1-\omega'}(\boldsymbol{\varepsilon}_1\cdot\boldsymbol{\varepsilon}')(\boldsymbol{\varepsilon}_2\cdot\boldsymbol{\varepsilon}')\times\Big(1+\frac{\omega\omega'-\mathbf{k}_1\cdot\mathbf{k}'}{m(\omega_1-\omega')}+\cdots\Big)=\frac{e^4}{(2\pi)^3}\frac{1}{m}\frac{8\pi}{3}(\boldsymbol{\varepsilon}_1\cdot\boldsymbol{\varepsilon}_2)\Big(\omega_1\ln\omega_1-2\frac{\omega_1{}^2}{m}\ln\omega_1\Big).\quad(5.7)$$

For the second and the third terms in Eq. (5.6) the recoil factors can be dropped; after a simple computation, the result is

$$\frac{e^4}{3\pi^2}\frac{(\boldsymbol{\varepsilon}_2\cdot\mathbf{k}_1)(\boldsymbol{\varepsilon}_1\cdot\mathbf{k}_2)}{m^2}\ln\omega_1.\quad(5.8)$$

The spin-dependent terms will give a term of the order $\omega_1{}^2\ln\omega_1$. When the contribution from the crossed graphs is added, the leading $\omega_1\ln\omega_1$ term and the spin-dependent terms are both canceled by the cross terms. By Eqs. (5.2), (5.7), (5.8), and (5.1), $C^{(4)}$ is given as

$$C^{(4)}=-\frac{e^2}{3\pi^2}\Big\{\Big[\Big(\frac{\omega}{m}\Big)^2(3+\cos\Theta)(\boldsymbol{\varepsilon}_1\cdot\boldsymbol{\varepsilon}_2)-\frac{2(\boldsymbol{\varepsilon}_1\cdot\mathbf{k}_2)(\boldsymbol{\varepsilon}_2\cdot\mathbf{k}_1)}{m^2}\Big]\times\ln\omega+\Big(\frac{\omega}{m}\Big)^2(1-\cos\Theta)(\boldsymbol{\varepsilon}_1\cdot\boldsymbol{\varepsilon}_2)\int\frac{d\omega'}{\omega'}\Big\}.\quad(5.9)$$

The cross product of the e^4 amplitude and the Thomson amplitude $2e^2\boldsymbol{\varepsilon}_1\cdot\boldsymbol{\varepsilon}_2$ gives rise to the leading e^6 differential cross section

$$d\sigma(6)=d\Omega\frac{2\alpha}{3\pi}\Big(\frac{\alpha}{m}\Big)^2\Big(\frac{\omega}{m}\Big)^2\Big((\cos^3\Theta-3\cos^2\Theta-3\cos\Theta-3)\times\ln\omega-(1-\cos\Theta)(1+\cos^2\Theta)\int\frac{d\omega'}{\omega'}\Big)+O(\omega^2).\quad(5.10)$$

This result agrees with the low-energy limits of the exact relativistic results calculated in perturbation theory by Corinaldesi and Jost for spin-zero targets, and by Brown and Feynman for electron targets.[28]

With our approach the spin-independent nature of the results in Eqs. (5.9) and (5.10) are understood[29] and, furthermore, they are shown to be exact in strong interactions.

Remarks. One may ask why the e^2 radiative correction invalidates the ω^0 bremsstrahlung low-energy theorem, but not the ω Compton theorem. The reason lies essentially in the special crossing properties of the Compton scattering. In bremsstrahlung, the momenta of the initial and final charged particles are not correlated in the soft-photon limit; on the other hand, in Compton scattering we have the kinematic relation $k_1\cdot(p_1-p_2)=k_1\cdot k_2$. Accordingly, the leading $\omega\ln\omega$ term [see Eq. (5.7)] is canceled when the corresponding contribution from the crossed graphs is added.[30] It can be understood in a similar way that the $\omega\ln\omega$ amplitude vanishes for π^0 photoproduction.

[28] F. Corinaldesi and R. Jost, Helv. Phys. Acta **21**, 183 (1948); L. M. Brown and R. P. Feynman, Phys. Rev. **85**, 231 (1952); see also W. Heitler and S. T. Ma, Phil. Mag. **11**, 651 (1949).

[29] That in the perturbation calculation the result of the e^6 Compton cross section for spin $\frac{1}{2}$ should be identical in the low-frequency limit to that for a spin-0 target was pointed out as a puzzle by W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, New York, 1954), 3rd ed., p. 332.

[30] This apparently the source of an error in the calculation of the low-energy e^6 Compton total cross section by Soloviev (Ref. 7, p. 667). Evidently this author kept only terms corresponding to the leading $\omega\ln\omega$ factor of Eq. (5.7) (by analogy with the bremsstrahlung case) and arrived at the value $\frac{4}{3}(\alpha/\pi)(\omega/m)^2(\ln\omega)\sigma_0$ instead of the correct value $-4(\alpha/\pi)(\omega/m)^2(\ln\omega)\sigma_0$, where $\sigma_0=(8\pi/3)(\alpha/m)^2$.

ACKNOWLEDGMENTS

It is indeed a pleasure to thank Professor A. Pais and Professor M. A. B. Bég for their continual encouragement and advice.

APPENDIX

Equation (4.4) offers us a general procedure for evaluating the nonanalytic threshold factors. Here we will derive it via the noncovariant Low equation. We shall again consider the radiative corrections to the process $p_1+r_1 \to p_2+r_2+k$. The vectorial amplitude M_λ is related to the matrix element of the current operators by

$$M_\lambda(r_2,p_2,k\,;r_1,p_1)=-(2\pi)^3(4E_1E_2)^{1/2}\int d^4x d^4y \times e^{-ir_2x+ir_1y}\langle \mathbf{p}_2|T(j(x)J_\lambda(0)j(y))|\mathbf{p}_1\rangle. \quad \text{(A1)}$$

E_1 (E_2) is the energy variable of p_1 (p_2). J_λ is the electromagnetic current operator and the j's are the source currents for the initial and final neutral particles. There may be, in addition, equal-time commutator factors, which precisely compensate the noncovariant nature of the T product. Since we are only interested in the singular threshold contributions of soft-photon intermediate states, we can ignore these factors, since they are not singular in the limit $\omega \to 0$.

First, we write out the time-ordered product

$$\begin{aligned}T(j(x)J_\lambda(0)j(y)) &= \theta(x_0)\theta(-y_0)j(x)J_\lambda(0)j(y)+\theta(y_0)\theta(-x_0)j(y) \\ &\quad J_\lambda(0)j(x)+\theta(-x_0)\theta(x_0-y_0)J_\lambda(0)j(x)j(y) \\ &\quad +\theta(-y_0)\theta(y_0-x_0)J_\lambda(0)j(y)j(x) \\ &\quad +\theta(x_0-y_0)\theta(y_0)j(x)j(y)J_\lambda(0) \\ &\quad +\theta(y_0-x_0)\theta(x_0)j(y)j(x)J_\lambda(0). \quad \text{(A2)}\end{aligned}$$

It is not difficult to check that only the last four terms contain amplitudes having energy denominators that may vanish in the zero-frequency limit. Consider the contribution coming from the third term in Eq. (A2):

$$\sum_n \int d^4x d^4y\, e^{-ir_2\cdot x+ir_1\cdot y}\theta(-x_0)\theta(x_0-y_0) \times\langle \mathbf{p}_2|J_\lambda(0)|\mathbf{n}\rangle\langle\mathbf{n}|j(x)j(y)|\mathbf{p}_1\rangle, \quad \text{(A3)}$$

where $|\mathbf{n}\rangle$ denotes a general on-mass-shell intermediate state with total three-momentum $\mathbf{n}$ and energy E_n. By translational invariance and appropriate change of space-time variables, we have

$$\begin{aligned}&\sum_n \int d^4x d^4y\, e^{-i(n-p_2-k)y+i(n-p_1-r_1)x}\theta(-y_0)\theta(x_0) \times\langle \mathbf{p}_2|J_\lambda(0)|\mathbf{n}\rangle\langle\mathbf{n}|j(x)j(0)|\mathbf{p}_1\rangle \\ &=\sum_n (2\pi)^3\delta(\mathbf{n}-\mathbf{p}_2-\mathbf{k})\frac{1}{\omega+E_2-E_n}i\langle\mathbf{p}_2|J_\lambda(0)|\mathbf{n}\rangle \times\int d^4x\, e^{i(n-p_1-r_1)x}\theta(x_0)\langle\mathbf{n}|j(x)j(0)|\mathbf{p}_1\rangle. \quad \text{(A4)}\end{aligned}$$

The fourth term in Eq. (A2) may be written in a similar way, and when it is combined with Eq. (A4) the result is

$$-(2\pi)^6(4E_1E_2)^{1/2}\sum_n \delta(\mathbf{n}-\mathbf{p}_2-\mathbf{k}) \times\frac{\langle\mathbf{p}_2|J_\lambda|\mathbf{n}\rangle\langle\mathbf{n},\mathbf{r}_2|M|\mathbf{p}_1,\mathbf{r}_1\rangle}{E_n-E_2-\omega}, \quad \text{(A5)}$$

where

$$\langle\mathbf{n},\mathbf{r}_2|M|\mathbf{p}_1,\mathbf{r}_1\rangle \equiv -i\int d^4x\, e^{-ir_2x}[\theta(-x_0)\langle\mathbf{n}|j(0)j(x)|\mathbf{p}_1\rangle +\theta(x_0)\langle\mathbf{n}|j(x)j(0)|\mathbf{p}_1\rangle e^{i(E_n-E_2-\omega)x_0}].$$

For singular terms, the off-shell effects in the numerator may be neglected. Equation (4.4) then corresponds to the contribution by the state of a particle plus a soft photon, $\mathbf{n}=\mathbf{p}'+\mathbf{k}'$ and $E_n=E'+\omega'$, with $E'\equiv(\mathbf{p}'^2+m^2)^{1/2}$, to the sum in Eq. (A5). We note the following correspondences:

$$\sum_n \leftrightarrow \int d\mathbf{p}'d\mathbf{k}'\sum_{\epsilon'},$$

$$2E'(E'+\omega'-E_2-\omega)\leftrightarrow p'^2+m^2,$$

$$\langle\mathbf{p}_2|J_\lambda|\mathbf{p}',\mathbf{k}'(\boldsymbol{\varepsilon}')\rangle[(2\pi)^9 8E_2E'\omega']^{1/2}\leftrightarrow \epsilon_\mu' C_{\lambda\mu}(k,p_2\,;k',p'),$$

$$\langle\mathbf{r}_2,\mathbf{p}',\mathbf{k}'(\epsilon')|M|\mathbf{p}_1\mathbf{r}_1\rangle[(2\pi)^9 8E_1E'\omega']^{1/2} \leftrightarrow \epsilon_\nu' M_\nu(k',p'\,;p_1).$$

The contributions corresponding to the fifth and sixth terms in Eq. (A2) are just the crossed graphs.