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# Higgs phenomena in asymptotically free gauge theories*

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We examine in detail the possibility of using the Higgs mechanism to remove the catastrophic infrared singularities in non-Abelian gauge theories which are asymptotically free. Our investigation encompasses theories based on SU($N$) or O($N$) with scalars in one vector, two vector, $M$ vector, adjoint, tensor, and adjoint plus one vector representations. We find that for these theories an $S$ matrix, in the perturbative sense, and asymptotic freedom can not coexist. We show that a wide class of Yukawa couplings can be ignored in studying the large-momentum properties of the scalar couplings.

## I. INTRODUCTION

Recently, Politzer,[1] and Gross and Wilczek[2] have discovered,[3] using the Gell-Mann–Low renormalization-group techniques,[4] that for non-Abelian gauge theories the origin of the coupling-constant space is a stable fixed point[5] in the deep Euclidean limit. Theories having this property are now often referred to in the literature as being "asymptotically free."[6] Furthermore, extending the result of an earlier work of Zee,[7] Coleman and Gross[8] have shown that no renormalizable field theory can be asymptotically free without non-Abelian gauge fields. There are also theoretical arguments[9] indicating that Bjorken scaling as observed in the deep-inelastic electroproduction experiments at SLAC and its implied canonical behavior of the light cone in the configuration space can only be obtained in a theory where the effective coupling constants vanish in this asymptotic limit. All these developments lead us to the following conclusion: If Bjorken scaling, as a true asymptotic phenomena,[10] is to be explained in renormalizable field theories, we must have a non-Abelian Yang-Mills theory for the strong interactions.

The great practical advantages of asymptotically free gauge theories is that one can study certain physically interesting strong-interaction quantities in the ultraviolet regime by perturbative methods. The high-energy $e^+e^-$ annihilation (via one photon) total cross section goes as $s^{-1}$ with calculable logarithmic corrections.[11] The anomalous dimensions of the entire tower of low-twist operators in the Wilson light-cone expansion may also be calculated.[12] It was found that such theories give scaling up to logarithms.[13-15]

One of the difficulties of asymptotically free gauge theories is the presence of very severe infrared singularities coming from the massless nature of Yang-Mills particles. Because of the non-Abelian nature of these gauge symmetries, these infrared singularities cannot be handled like those of the usual quantum electrodynamics (QED).[16] The consequence is that only off-shell Green's functions can be studied, and the on-shell $S$-matrix elements are left undefined. However, it is well known that the Higgs phenomenon,[17] in which the gauge symmetry breaks spontaneously, can give masses to the gauge particles in such a way that the renormalizability of the theory is preserved. The aim of this paper is to investigate whether by this mechanism we can remove all the infrared singularities while maintaining the asymptotic freedom.

The problem of incorporating scalars in the asymptotically free theories was first examined by Gross and Wilczek.[13] The difficulty, as pointed out by these authors, lies in the fact that the self-quartic couplings of scalars, inevitably present in any renormalizable field theories involving scalars, are inherently unstable but for the presence of gauge couplings. In Ref. 13, scalars are restricted to a single low-dimensional representation. The cases of one vector, one adjoint, or one symmetric tensor in SU($N$) and the case of scalars belonging to one $(N, N)$ in SU($N$) $\times$ SU($N$) are worked out. Furthermore, Yukawa couplings are excluded. Since Yukawa couplings enter in a nontrivial way in the renormalization-group differential equations of scalar couplings [see Eq. (2.8)], they can in principle have an effect on the stability properties of the scalar couplings.

In Sec. II we set up the general formalism and establish our notation. We make some general remarks about the calculation of the coefficients in the renormalization-group equations for the coupling constants and discuss our method of determining the stability of this system of differential equations. We also briefly discuss the situation in semisimple groups. We note that if any one of the product groups is a U(1) the theory will not be asymptotically free.

In Sec. III we consider the effect of the presence of Yukawa couplings on the asymptotic freedom of the theory. We demonstrate that if the Yukawa couplings are stable at zero they will be driven to zero at a faster rate than the gauge coupling constant itself and hence will not influence the stability properties of the scalar couplings. The proof for a case of more than one Yukawa coupling is left to Appendix A.

In Sec. IV we investigate the asymptotic stability of the scalar coupling constants. We examine scalars belonging to one vector, two vector, $M$ vector, adjoint, antisymmetric tensor, symmetric tensor, and adjoint plus one vector representations of O($N$) and SU($N$). Many of the details of the calculations are left to Appendix B. We observe a general pattern of the occurrence of asymptotic freedom in gauges theories with scalars. As the number of scalar fields increases, the dimension of the group and therefore the number of gauge bosons must also be increased to maintain asymptotic freedom. We find no examples when the symmetry is broken down to an Abelian symmetry which is also ultraviolet-stable. For completeness we include an Appendix C that reviews the results of one of us (L.-F.L.)[18] on the symmetry breaking induced by the Higgs mechanism for various representations of SU($N$) and O($N$).

In Sec. V we make some concluding remarks as to the compatibility of asymptotic freedom and the existence of an $S$ matrix in perturbation theory.

## II. GENERAL FORMALISM

The most powerful technique used in studying the asymptotic properties of the renormalizable field theory is the Callan-Symanzik equations. We refer the reader to Coleman's Erice lecture[19] for an elegant presentation of this technique. Here we write down only those differential equations for the invariant coupling constants which are needed for studying whether the gauge theories can be asymptotically free in the presence of the Higgs phenomenon.

Let $A_\mu^a$, $\phi_i$, and $\psi_\alpha$ be the Hermitian gauge fields, real scalar fields, and spin-$\frac{1}{2}$ fields, respectively. The most general renormalizable Lagrangian which is locally gauge-invariant under some simple compact Lie group $\mathcal{G}$ with structure constants $C^{abc}$ is as follows:

$$\mathcal{L} = -\tfrac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \tfrac{1}{2}(D_\mu\phi)_i (D^\mu\phi)_i + i\bar\psi\gamma^\mu D_\mu\psi - \bar\psi m_0\psi - \bar\psi h_i\psi\phi_i - V(\phi), \tag{2.1}$$

where $V(\phi)$ is some quartic polynomial in $\phi$,

$$V(\phi) = \frac{1}{4!} f_{ijkl}\,\phi_i\phi_j\phi_k\phi_l + \text{lower-order terms}, \tag{2.2}$$

and where

$$F^a_{\mu\nu} = \partial_\nu A^a_\mu - \partial_\mu A^a_\nu - g C^{abc} A^b_\mu A^c_\nu, \tag{2.3}$$

$$(D_\mu\phi)_i = \partial_\mu\phi_i + i g\theta^a_{ij}\phi_j A^a_\mu, \tag{2.4}$$

$$(D_\mu\psi)_\alpha = \partial_\mu\psi_\alpha + i g t^a_{\alpha\beta}\psi_\beta A^a_\mu. \tag{2.5}$$

The scalar (fermion) fields belong to some, in general reducible, representation of $\mathcal{G}$, with corresponding representation matrices of the generators, $\theta^a_{ij}$ ($t^a_{\alpha\beta}$). The constraints on $\theta^a$, $t^a$, $m_0$, $h$, and $V$ due to requirements of hermiticity and gauge invariance of the Lagrangian may be easily worked out.

Given the above Lagrangian we can immediately write down the lowest-order approximation of the renormalization-group equation for the effective coupling constants:

$$16\pi^2\frac{dg}{dt} = -\left[\tfrac{11}{3}S_1(\mathcal{G}) - \tfrac{4}{3}S_3(F) - \tfrac{1}{6}S_3(S)\right]g^3 \equiv -\tfrac{1}{2}b_0 g^3, \tag{2.6}$$

$$16\pi^2\frac{dh_i}{dt} = 2h_m h_i h_m + \tfrac{1}{2}(h_m h_m h_i + h_i h_m h_m) + 2\,\mathrm{Tr}(h_i h_m)h_m - 3[2t^a h_i t^a + S_2(F)h_i]g^2 + \frac{1}{288\pi^2} f_{ijkl} f_{jklm} h_m, \tag{2.7}$$

$$16\pi^2\frac{df_{ijkl}}{dt} = f_{ijmn}f_{mnkl} + f_{ikmn}f_{mnjl} + f_{ilmn}f_{mnjk} - 12S_2(S)g^2 f_{ijkl} + 3A_{ijkl}\,g^4 + 8\,\mathrm{Tr}(h_i h_m) f_{mjkl} - 12H_{ijkl}, \tag{2.8}$$

where

$$A_{ijkl} \equiv \{\theta^a, \theta^b\}_{ij}\{\theta^a, \theta^b\}_{kl} + \{\theta^a, \theta^b\}_{ik}\{\theta^a, \theta^b\}_{jl} + \{\theta^a, \theta^b\}_{il}\{\theta^a, \theta^b\}_{jk} \tag{2.9}$$

and

$$H_{ijkl} \equiv \frac{1}{3!}\mathrm{Tr}(h_i h_j\{h_k, h_l\} + h_i h_k\{h_j, h_l\} + h_i h_l\{h_j, h_k\}).$$

We have used the following identities:

$$
\begin{aligned}
&C^{acd}C^{bcd}=S_1(\mathcal{G})\,\delta^{ab}\,,\\
&(t^a t^a)_{ij}=S_2(F)\delta_{ij}\,,\\
&(\theta^a\theta^a)_{ij}=S_2(S)\delta_{ij}\,,\\
&\mathrm{Tr}(t^a t^b)=S_3(F)\delta^{ab}\,,\\
&\mathrm{Tr}(\theta^a\theta^b)=S_3(S)\delta^{ab}\,,
\end{aligned}
\qquad (2.10)
$$

which define constants $S_1$, $S_2$, and $S_3$ depending only on the group ($\mathcal{G}$) and the representation of the fermions ($F$) and scalars ($S$). We have also used the fact that $f_{ijkl}$ is totally symmetric.

A number of comments concerning this set of coupled equations are in order.

(a) These equations apply in the deep Euclidean region where all the dimensional coupling constants, i.e., mass terms and superrenormalizable interactions, can be dropped. They describe the response of the coupling constants when the normalization point $M$ at which the couplings are defined is changed to $\lambda M$. Thus the coupling constants appearing in the above equations are functions of $t\equiv -\ln\lambda$ and are commonly referred to as "effective coupling constants" or "running coupling constants." The boundary condition for these effective coupling constants is that they are equal to the couplings appearing in the Lagrangian defined at Euclidean symmetric points $p^2=-M^2$. For simplicity of notation we shall not give the effective coupling constants separate labels, and their dependence on $t$ is to be understood in the following discussion.

(b) Since we are only interested in exploring around the origin of coupling constant space, it is adequate for our purposes to calculate to lowest order. It should be noted that the lowest-order terms are not necessarily all single-loop diagrams. In particular, Fig. 2(c) is a two-loop term. However, as it turns out, the structure of Eq. (2.8) is such that if the theory is to be asymptotically free the scalar couplings $f$ must be proportional to $g^2$ as $g$ approaches zero in the ultraviolet limit. Consequently, the $hff$ term may be dropped and only one-loop diagrams need be considered.

(c) The algorithm for the perturbation calculation of the right-hand side of Eqs. (2.6)–(2.8) is by now well known. It involves calculating the logarithmically divergent parts of diagrams which contribute to the coupling-constant renormalization, and then taking the logarithmic derivative with respect to mass scale $M$. These lowest-order equations are gauge-independent.[13] The computations are simplest when done in the Landau gauge where many one-loop diagrams are finite and hence do not contribute. The relevant diagrams for Eqs. (2.6), (2.7), and (2.8) are displayed

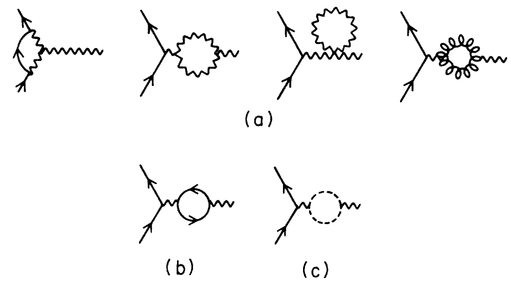


FIG. 1. Lowest-order ($g^3$) contributions to the gauge-coupling renormalization constant defined here from the vector-fermion vertex. The solid directed lines represent fermions, the wavy lines represent the gauge bosons, dashed lines represent scalars, and looped lines represent the Faddeev-Popov ghosts. (a), (b), and (c) correspond to the first, second, and third terms on the right-hand side of Eq. (2.6). Other possible one-loop diagrams not displayed do not contribute in the Landau gauge.

in Figs. 1, 2, and 3, respectively.

(d) We note that in the presence of spontaneous symmetry breaking via the Higgs mechanism the perturbation calculations of the renormalization-group equations for $g$, $h_i$, and $f_{ijkl}$ do not depend on whether or not the scalar fields are shifted to have zero vacuum expectation values, thus giving masses to the gauge bosons and fermions. This is because shifting the scalar fields can only affect superrenormalizable interaction terms in the Lagrangian; these terms do not affect the renormalization-group equations.

(e) As explained in Ref. 19, the renormalization-group equation for a single coupling constant, $f$, is of the form $df/dt=\beta(f)$ and has a stable fixed point at $f=0$ if $\beta(f)|_{f=0}=0$ and $d\beta/df|_{f=0}<0$. We first observe that Eq. (2.6) can be solved trivially to give

$$
g^2(t)=\frac{g^2(0)}{1+(b_0/16\pi^2)g^2(0)t}\,. \qquad (2.11)
$$

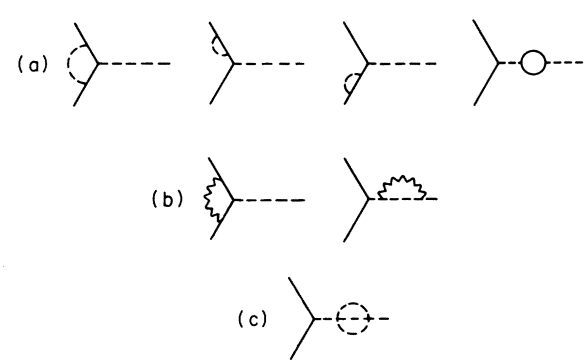


FIG. 2. Lowest-order contributions to the Yukawa-coupling renormalization constants. (a), (b), and (c) correspond to the $h^3$, $hg^2$, and $hf^2$ terms in Eq. (2.7).

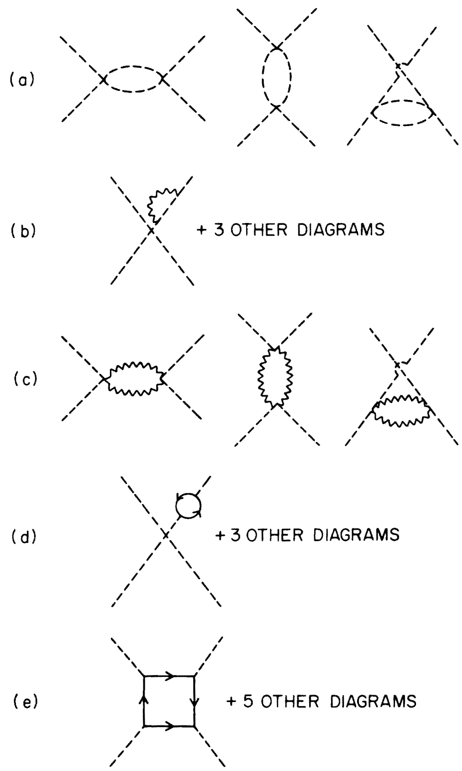


FIG. 3. Lowest-order contributions to the quartic self-coupling renormalization constants. (a), (b), (c), (d), and (e) correspond to the $f^2$, $fg^2$, $g^4$, $fh^2$, and $h^4$ terms in Eq. (2.8).

Hence if $b_0>0$, $g(t)$ is real and tends to zero as $t\to+\infty$. Equation (2.7) will be discussed in the next section; we will see that, in the cases of interest, if $h_i=0$ is a stable point as $t\to\infty$ then $\bar{h}_i=h_i/g$ goes to zero as $t\to\infty$. Therefore the Yukawa terms will be ignored in the equation for the scalar couplings, Eq. (2.8). In order to find the stability properties of Eq. (2.8) we eliminate the $g^2(t)$ dependence by introducing a variable $u\equiv(16\pi^2/b_0)\ln[g^{-2}(0)+(b_0/16\pi^2)t]$ and $\bar{f}_{ijkl}\equiv f_{ijkl}/g^2$. The stability equations for $\bar{f}_{ijkl}(u)$ may be written in the form

$$\frac{dx_i}{du}=F_i(\vec{x}), \tag{2.12}$$

where $\vec{x}$ is a column vector with components the $\bar{f}_{ijkl}$'s, and $F_i(\vec{x})$ are quadratic functions of the components of $\vec{x}$. We find the stable fixed points of this system of differential equations by the following method. We first find the solutions of the system of quadratic equations $F_i(\vec{x})=0\ \forall\, i$. The real solutions of $F_i(\vec{x})=0\ \forall\, i$ are called critical points of the differential equations. We now consider the stability of each of these solutions. We define the slope matrix of $F_i(\vec{x})$ at $\vec{x}_0$ by

$$D_{ij}(\vec{x}_0)\equiv\left.\frac{\partial F_i(\vec{x})}{\partial x_j}\right|_{\vec{x}=\vec{x}_0}. \tag{2.13}$$

The point $\vec{x}_0$ is a stable fixed point if (1) $F_i(\vec{x}_0)=0\ \forall i$ and (2) all the eigenvalues of $D_{ij}(\vec{x}_0)$ have negative real parts.

The simple way to understand these conditions is as follows. The local stability of the critical point $\vec{x}_0$ is determined by retaining in $F_i(x)$ only the linear term in $\vec{\xi}=\vec{x}-\vec{x}_0$;

$$\frac{d\xi_i}{du}=\sum_{j=1}^{n}D_{ij}(\vec{x}_0)\xi_j. \tag{2.14}$$

Using the standard method, we write

$$\xi_i=C_ie^{\lambda u} \tag{2.15}$$

to get the algebraic equations

$$\lambda C_i=D_{ij}(\vec{x}_0)C_j. \tag{2.16}$$

To have nontrivial solutions, we have to demand $\det(D(\vec{x}_0)-\lambda I)=0$, which gives $n$ complex eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. It is then easy to see from Eq. (2.15) that the critical point $x_0$ is locally stable if all the $\lambda_i$'s have negative real parts so that $\xi_i\to0$, or $\vec{x}\to\vec{x}_0$ as $u\to\infty$.[20]

We note that $\vec{x}_0$ being a stable fixed point implies that $f_{ijkl}\simeq g^2(t)\vec{x}_0$ goes to zero as $t\to\infty$. In most of the cases we consider in Sec. IV we use a computer to do these straightforward but tedious calculations.

(f) Up to this point we restricted our considerations to simple Lie groups, i.e., theories with only one gauge coupling constant. We shall make a brief comment on the more general cases involving semisimple groups, which are just direct products of simple groups $G_1\times G_2\times\cdots\times G_n$, each with its own coupling constant $g_i$. To lowest order ($g_i^3$) the renormalization-group equations for each of the coupling constants is independent of other coupling constants,[21] and therefore the results can be deduced from that of simple groups. It is interesting to note that if one of these groups, $G_i$, is an Abelian [U(1)] group, the associated gauge coupling will not be driven to zero and the theory is *not* asymptotically free. Since most unified theories of weak and electromagnetic interactions are based on the gauge group SU(2)×U(1), the ultraviolet behavior of such theories is not controlled by the fixed point at zero coupling and the Johnson-Baker-Willey and Adler programs remain a possible approach to QED within these unified theories.[22]

## III. RENORMALIZATION-GROUP EQUATION FOR THE YUKAWA COUPLINGS

No renormalizable field theory can be asymptotically free without non-Abelian gauge fields.

It is understood in this statement that one accepts the semiclassical arguments that the quartic self-coupling $f_{ijkl}$ must be a positive-definite tensor for the energy spectrum to be bounded below. On the other hand, Symanzik has suggested that such an argument may not be so compelling in a full quantum field theory[23] (i.e., beyond the tree-graph approximation). Clearly, if this positivity condition is relaxed, one can have an asymptotic free $\varphi^4$ theory. However, it should be pointed out that in the view of the works by Zee[7] and Coleman and Gross[8] it is impossible to construct any realistic theory even if the sign of $\varphi^4$ term can be changed. The nontrivial introduction of fermions will immediately destroy the asymptotic freedom since Yukawa couplings are inherently ultraviolet-unstable, independent of the choice of sign for the $\varphi^4$ term.

The asymptotic stability property of Yukawa couplings is altered dramatically in a non-Abelian gauge theory. As we shall see, Yukawa couplings can now be driven to zero in the deep Euclidean limit. In fact, they are driven to zero at a much faster rate than the gauge coupling itself.

To illustrate our point, consider first the simple solution where only one Yukawa coupling constant is present. The set of coupled equations in (2.7) is reduced to (recall that the $hf^2$ term is negligibly small)

$$16\pi^2 \frac{dh}{dt} = Ah^3 - Bg^2 h, \tag{3.1}$$

where $A$ and $B$ are some positive constants determined by the representation content of the fermions and scalars. For example, the case where scalars belong to the adjoint representation is just the generalization of the Yukawa coupling scheme of Ref. 7 to the case of gauge theories.[24] To solve this equation it is convenient to use the variable $\bar{h} = (h/g)^2$ to rewrite it in the form

$$8\pi^2 \frac{d\bar{h}}{du} = \bar{h}[A\bar{h} - (B - \tfrac{1}{2}b_0)], \tag{3.2}$$

where $u$ and $b_0$ are defined by Eqs. (2.12) and (2.6), respectively. If $B - \frac{1}{2}b_0 < 0$, there is only one critical point $\bar{h} = 0$, because $\bar{h} = (h/g)^2$ is a positive quantity. This case is unstable because the slope at this point is positive. In the case $B - \frac{1}{2}b_0 > 0$, there are two critical points, $\bar{h} = 0$ and $\bar{h} = (B - \frac{1}{2}b_0)/A$. The stable fixed point is at $\bar{h} = 0$ because the slope at this point is given by $-(B - \frac{1}{2}b_0) < 0$. This shows that $h$ must approach zero at a faster rate than $g$ if $B - \frac{1}{2}b_0 > 0$.

We now turn to cases where there are more than one Yukawa coupling. We consider the situation in which all the fermions belong to the fundamental representation of the group. Consequently, the only type of Yukawa couplings is the "vector-vector-adjoint" type, but the coupling constants for fermions in different fundamental representations need not be the same. For definiteness, let us say there are $M$ such sets. The scalars may belong to a reducible representation, but we stipulate that there is only one adjoint representation. Thus there are $\frac{1}{2}M(M+1)$ Yukawa couplings. We label them as follows:

$$h = h_{\alpha\beta} t^a, \tag{3.3}$$

where $t^a$, as before, is the representation matrix of the generators. The index $a$ runs from 1 to the order of the group. The indices $\alpha$ and $\beta$ denote the "fermion type," i.e., they distinguish the different sets of fermions. So we can view $h_{\alpha\beta}$ as an $M \times M$ symmetric coupling matrix. Writing the Yukawa couplings in this form, we have for Eq. (2.7)

$$16\pi^2 \frac{dh_{\alpha\beta}}{dt} = A(hhh)_{\alpha\beta} + \mathrm{Tr}(hh)h_{\alpha\beta} - Bg^2 h_{\alpha\beta}. \tag{3.4}$$

$A$ and $B$ are again positive constants. For example, in SU($N$), $A = (N^2-3)/2N$ and $B = 3(N^2-1)/N$. The stability properties of this set of coupled equations can be determined after they are simplified by diagonalization of the $h_{\alpha\beta}$ matrix. Again, we find (see Appendix A) that if the theory is asymptotically free, i.e., $h_{\alpha\beta} \to 0$, then $h_{\alpha\beta}/g \to 0$.

The equations for the effective scalar self-coupling constants, Eq. (2.8), are of the form

$$\frac{df}{dt} = af^2 + bfg^2 + cg^4 + dfh^2 + eh^4. \tag{3.5}$$

Since $h$ goes to zero faster than $g$, i.e. $h/g \to 0$ in the large ultraviolet regime, the $fh^2$ and $h^4$ terms clearly can be dropped when compared with $fg^2$ and $g^4$ terms.

## IV. RENORMALIZATION-GROUP EQUATIONS FOR THE SCALAR COUPLINGS

In this section we investigate the large-momentum behavior of the scalar couplings for those classes of gauge theories based on the familiar O($N$) and SU($N$) groups with various choices of the representation of the scalars up to second-rank tensors. Tensors higher than second rank are more difficult to handle. But we note that in Eq. (2.6) for the gauge coupling constant the contributions of gauge particles $S_1(\mathcal{G})$ have the right sign for asymptotic freedom and are proportional to $N$ in both O($N$) and SU($N$), while the scalar contributions $S_3(S)$ have the wrong sign and are proportional to $(N)^{k-1}$ for scalars belonging to $k$th-rank tensors (see Table III). For large enough $N$, any tensor higher than second rank will destabilize the gauge coupling and can be ruled out for our purposes.

Since the calculations are very similar in all cases, we will give only the results here and will give some of the details in Appendix B. We will first discuss the situation in O($N$) in Sec. IV A and will describe briefly the similar situation in SU($N$) in Sec. IV B, followed by a discussion in Sec. IV C.

### A. Stabilities of scalar couplings in O($N$)

We first discuss the simplest case with only one vector representation in O($N$). The most general O($N$)-invariant quartic coupling contains only one coupling constant;

$$\mathcal{L}_I = -\frac{\lambda}{8}\sum_{a=1}^{N}(\phi_a\phi_a)^2 . \tag{4.1}$$

The renormalization-group equation for $\lambda$ takes the form

$$\frac{d\lambda}{dt} = \frac{1}{16\pi^2}\left[(N+8)\lambda^2 - 3(N-1)\lambda g^2 + \tfrac{3}{4}(N-1)g^4\right], \tag{4.2}$$

or in terms of new variable $\bar\lambda = \lambda/g^2$

$$\begin{aligned}\frac{1}{g^2}\frac{d\bar\lambda}{dt} &= \beta(\bar\lambda)\\ &= \frac{1}{16\pi^2}\left\{(N+8)\bar\lambda^2 + \left[b_0 - 3(N-1)\right]\bar\lambda + \tfrac{3}{4}(N-1)\right\},\end{aligned} \tag{4.3}$$

where $b_0$ is defined by Eq. (2.6),

$$\frac{dg}{dt} = -b_0 g^3\left(\frac{1}{32\pi^2}\right), \quad \text{and } b_0 > 0 . \tag{4.4}$$

Since the right-hand side of Eq. (3.3) is a second-order polynomial of the form $\beta(\bar\lambda) = A\bar\lambda^2 + B\bar\lambda + C$, the condition for $\beta(\bar\lambda) = 0$ to have real roots is simply $\Delta = B^2 - 4AC \geq 0$. Let us call these two roots $\lambda_1$, $\lambda_2$ with $\lambda_2 > \lambda_1$ and calculate the slope at these points:

$$\left.\frac{d\beta(\lambda)}{d\lambda}\right|_{\lambda=\lambda_1} = A(\lambda_1 - \lambda_2) < 0 ,$$

$$\left.\frac{d\beta(\lambda)}{d\lambda}\right|_{\lambda=\lambda_2} = A(\lambda_2 - \lambda_1) > 0 .$$

Hence the smaller root $\lambda_1$ is a stable fixed point. However, from the condition that the classical potential corresponding to this interaction is bounded below we have to require $\lambda > 0$. This constraint demands $B < 0$ because both $A$ and $C$ are positive. This implies that it is most favorable to have $b_0$ as small as possible in order to have large $|B|$ to satisfy $\Delta = B^2 - 4AC \geq 0$. This can be achieved by having as many fermions as possible without changing the sign of $b_0$. It turns out that in all the cases we consider the results do not change very much if we use the smallest possible $b_0$ instead of $b_0 = 0$. So for the sake of simplicity we will assume from now on that $b_0 = 0$. Then the discriminant condition is simply

$$3(N-1)(2N-11) > 0 . \tag{4.5}$$

The theory is stable for $N \geq 6$. On the other hand it has been shown that for one vector representation in O($N$), the symmetry is broken from O($N$) to O($N-1$) (see Appendix C). This means that only for $N = 2$ does it break the symmetry completely, and for O(6), the smallest group which has the asymptotic freedom, there are still ten massless gauge particles, corresponding to the generators of a *non-Abelian* O(5) group.

We next consider the case where we use two sets of scalars $\phi_a$ and $\psi_a$, each belonging to the vector representation of O($N$). The quartic scalar interaction contains four coupling constants:

$$-\mathcal{L}_{\text{int}} = \tfrac{1}{8}\lambda_1(\phi_a\phi_a)^2 + \tfrac{1}{8}\lambda_2(\psi_a\psi_a)^2 + \tfrac{1}{4}\lambda_3(\phi_a\phi_a)(\psi_b\psi_b) + \tfrac{1}{2}\lambda_4(\phi_a\psi_a)(\phi_b\psi_b) . \tag{4.6}$$

We have imposed a discrete symmetry $\phi_a \to -\phi_a$ to eliminate terms of the form $(\phi_a\psi_a)(\psi_b\psi_b)$ for simplicity. Notice that we have to deal with four coupling constants, as compared with only one in the previous case. The renormalization-group equations for these coupling constants are also very complicated:

$$\frac{d\lambda_1}{dt} = \frac{1}{16\pi^2}\left[(N+8)\lambda_1^2 + N\lambda_3^2 + 4\lambda_3\lambda_4 + 4\lambda_4^2 - 3(N-1)\lambda_1 g^2 + \tfrac{3}{4}(N-1)g^4\right], \tag{4.7a}$$

$$\frac{d\lambda_2}{dt} = \frac{1}{16\pi^2}\left[(N+8)\lambda_2^2 + N\lambda_3^2 + 4\lambda_3\lambda_4 + 4\lambda_4^2 - 3(N-1)\lambda_2 g^2 + \tfrac{3}{4}(N-1)g^4\right], \tag{4.7b}$$

$$\frac{d\lambda_3}{dt} = \frac{1}{16\pi^2}\left[(N+2)(\lambda_1+\lambda_2)\lambda_3 + 2(\lambda_1+\lambda_2)\lambda_4 + 2\lambda_3^2 + 2\lambda_4^2 - 3(N-1)\lambda_3 g^2 + \tfrac{3}{4}g^4\right], \tag{4.7c}$$

$$\frac{d\lambda_4}{dt} = \frac{1}{16\pi^2}\left[2(\lambda_1+\lambda_2)\lambda_4 + 4\lambda_3\lambda_4 + (N+2)\lambda_4^2 - 3(N-1)\lambda_4 g^2 + \tfrac{3}{8}(N-2)g^4\right]. \tag{4.7d}$$

We can proceed by using the variable $\bar{\lambda}_i = \lambda_i/g^2$ and finding all the real roots of these four simultaneous nonlinear equations, and then we can check the stability by calculating the eigenvalues of the slope matrix evaluated at these roots. Because of the complexity of all these calculations, we do all these steps numerically on the computer. The result is that this class of theories will become asymptotically free for $N \geq 7$. But the symmetry is broken from O($N$) to O($N-2$) by these two sets of vector representations (see Appendix C). Hence, for $N \geq 7$ the Higgs phenomenon fails to remove all the infrared singularities.

When we compare the two cases we have considered so far, it seems that as we put more scalars into the theory to break the symmetries further down, the threshold value of $N$ for asymptotic freedom of the theory also increases.

Since the number of coupling constants increases rapidly with the number of vector representations, it is very difficult to study the case with more than two vector representations, even on the computer. However, we observe that Eqs. (4.7a)–(4.7d) are invariant under $\lambda_1 \leftrightarrow \lambda_2$, which corresponds to $\phi_a \leftrightarrow \psi_a$ in the Lagrangian. The numerical solutions for these simultaneous equations have the property $\lambda_1 = \lambda_2$. If we impose the extra symmetry that the Lagrangian is invariant under $\phi_a \leftrightarrow \psi_a$, then it implies $\lambda_1 = \lambda_2$ and reduces the number of equations to three. In the case where we consider $M$ vector representations, $\phi^{(1)}_{a_1}$, $\phi^{(2)}_{a_2}, \ldots, \phi^{(M)}_{a_M}$, we can impose the interchange symmetry among these $M$ sets of scalar field to reduce the number of independent coupling constants to only three. The final results are summarized in Table I. This table also shows the pattern we mentioned before. In the limit $N$ is very large, the theory is stable for $M \leq 0.85N$, but the symmetry is broken completely for $M \geq N-1 \approx N$.

We now consider the more complicated second-rank tensor representations of O($N$). We give only the results here. Details of the quartic couplings

TABLE I. The threshold values for asymptotic freedom in O($N$) with various vector representations and the pattern of symmetry breaking.

| $M$ [a] | $N$ threshold for asymptotic freedom | Symmetry breaking |
|---|---|---|
| 3 | 7 | $O(N) \to O(N-3)$ |
| 4 | 8 | $O(N) \to O(N-4)$ |
| 5 | 9 | $O(N) \to O(N-5)$ |
| 6 | 10 | $O(N) \to O(N-6)$ |
| 7 | 11 | $O(N) \to O(N-7)$ |

[a] Number of vector representations.

TABLE II. The threshold values for asymptotic freedom in SU($N$) with various vector representations and the pattern of symmetry breaking.

| $M$ [a] | $N$ threshold for asymptotic freedom | Symmetry breaking |
|---|---|---|
| 3 | 5 | $SU(N) \to SU(N-3)$ |
| 4 | 6 | $SU(N) \to SU(N-4)$ |
| 5 | 7 | $SU(N) \to SU(N-5)$ |
| 6 | 8 | $SU(N) \to SU(N-6)$ |
| 7 | 10 | $SU(N) \to SU(N-7)$ |
| 8 | 11 | $SU(N) \to SU(N-8)$ |

[a] Number of vector representations.

and stability equations are given in Appendix B. For the second-rank symmetric tensor representation, the asymptotic freedom starts from $N = 14$. For the antisymmetric second-rank tensor representation, it is asymptotically free for $N \geq 8$. If we add a vector representation to this antisymmetric tensor, the threshold value of $N$ for asymptotic freedom starts to move up to $N = 9$. But for all these cases, it can be shown that the Higgs mechanism fails to remove the infrared singularities.

### B. Stabilities of scalar couplings in SU($N$)

The situation in SU($N$) is very similar to that in O($N$). We only describe the pattern here, and we refer the reader to Appendix B for the form of quartic couplings and stability equations.

In the simplest case of only one vector representation of SU($N$), the theory is asymptotically free for $N \geq 3$, and the symmetry-breaking pattern is from SU($N$) → SU($N-1$). Hence SU(3) gauge symmetry has asymptotic freedom and also has the infrared singularities associated with unbroken SU(2) symmetry. We next consider the case of two vector representations, which is enough to break the SU(3) symmetry completely. It turns out that the starting value for asymptotic freedom moves up to $n = 4$, and SU(4) is reduced only to SU(2) by two vector representations. For the cases with more than two vector representations, we impose the same interchange symmetries among these vector representations as in O($N$). The results are summarized in Table II. This shows the same pattern as we indicated in the case of O($N$). In the limit of large $N$ we can solve the equations analytically to show that it is unstable for $N-1$ vectors in SU($N$). We next examine the cases with second-rank tensor representations. The choices of second-rank symmetric tensor and adjoint representations have been studied by Gross and Wilczek.[13] We include their results here for completeness and comparison. In the case of the symmetric tensor representation, the theory becomes asymptotically free for $N \geq 9$. If we use only one adjoint representation, the asymptotic freedom will start from $N=6$. When we add one vector representation to the adjoint representation, the threshold for asymptotic freedom move up to $N=7$. We have also considered an antisymmetric tensor in SU($N$); we find asymptotic freedom for $N \geq 5$. Again for these values of $N$ there is always a non-Abelian symmetry left after the spontaneous symmetry breaking through the Higgs mechanism.

### C. Discussion

The general pattern for all these cases seems to be that when we put in scalars to remove all the infrared singularities the theory will lose the asymptotic freedom, or if we insist on asymptotic freedom the maximum sets of scalars we can put into the theory will only break the symmetry down to some non-Abelian symmetry. For example, in the most familiar SU(3) group, it is asymptotically free if we use only one triplet of scalars, but there is a non-Abelian SU(2) symmetry left unbroken, which leads to the uncontrollable infrared catastrophe. If we use two triplets of scalars to break the SU(3) symmetry completely, the theory will no longer be asymptotically free.

We have not exhausted all the possible choices of representations for the scalars. Since there are so many instances exhibiting the same feature, we conjecture this to be a very general property of the Higgs phenomenon in asymptotically free gauge theories. The validity of this conjecture and the possible physical mechanism responsible for this property are under further investigation.

The fact that asymptotic freedom is only possible in the non-Abelian gauge theories and not in any of the other renormalizable theories (e.g., Abelian gauge theories, Yukawa coupling theories and $\lambda\phi^4$ theories) is not well understood. It is a possibility that the asymptotic freedom on non-Abelian gauge theory is due to the presence of the infrared singularities, since there is no infrared catastrophe in any other theory.

## V. CONCLUSION

The systematic investigation of the effect of scalar couplings on a wide class of non-Abelian gauge theories leads to the results that the Higgs phenomenon fails to remove the infrared singularities in such a way that the asymptotic freedom is preserved. Whenever enough scalars are introduced to break the gauge symmetries completely, the theory loses its asymptotic freedom. This seems to indicate an intimate connection between the infrared singularities and the asymptotic freedom in the non-Abelian gauge theories. It deserves further investigations, which may lead to a better understanding of this feature; hopefully, progress in this direction will shed some light on the origin of the asymptotic freedom.

One might hope that somehow the symmetry is broken dynamically to give masses to the gauge particles instead of the simple Higgs mechanism. So far only the plausibility of this idea has been demonstrated in the context of Abelian gauge theory.[25] However, there is some difficulty in applying those arguments to the more interesting non-Abelian gauge theory. One of the crucial assumptions needed to demonstrate the possibility of having a spontaneously-broken-symmetry solution is that the fermion self-energy $\Sigma(p)$ has the asymptotic behavior $\Sigma(p) \sim (1/p^2)^{\gamma(g^2)}$, i.e., $\Sigma(p)$ has anomalous dimension. However, in the asymptotically free theory $\Sigma(p)$ behaves like $(\ln p^2)^a$, i.e., it can never have anomalous dimension.[26] Hence if this approach is to be workable at all, new techniques are needed to implement this idea.

In the absence of any reasonable physical mechanism to break the gauge symmetries to give masses to the gauge particles, it seems that we have to face the severe infrared singularities in the non-Abelian gauge theories seriously. One has to handle the infrared singularities one way or the other in order to have well-defined $S$-matrix elements. It has been speculated by a number of people that the local gauge symmetry may in fact remain exact and that the strong-coupling nature of the theory in the infrared limit provides the desired mechanism for quark confinement.[27] Although this possibility is very attractive, it has not been well formulated in any kind of field-theoretical framework. Undoubtedly any progress along these lines will lead to a tremendous enhancement in the understanding of strong-interaction physics.

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## APPENDIX A

In this appendix we shall demonstrate that the only possible asymptotic stable fixed point of the equation for $H_{\alpha\beta} = h_{\alpha\beta}/g$ [see Eq. (3.4)]

$$16\pi^2 \frac{dH_{\alpha\beta}}{du} = A(HHH)_{\alpha\beta} + \mathrm{Tr}(HH)H_{\alpha\beta} - (B - \tfrac{1}{2}b_0)H_{\alpha\beta} \tag{A1}$$

is $H_{\alpha\beta}(\infty)=0$.

Following the procedure for finding asymptotic stable fixed points outlined in Sec. II E, we must first find solutions to the coupled nonlinear equations

$$A(HHH)_{\alpha\beta}+\mathrm{Tr}(HH)H_{\alpha\beta}-B'H_{\alpha\beta}=0\,, \tag{A2}$$

where $B'=B-\frac{1}{2}b_0$. Then we must determine for each solution of (A2) whether or not all eigenvalues of the corresponding derivative matrix, Eq. (2.11), have negative real parts.

We shall solve Eq. (A2) by first diagonalizing the Hermitian matrix $H_{\alpha\beta}$. In terms of the diagonalized matrix $H'_{\alpha\beta}=\delta_{\alpha\beta}H_\alpha$, Eq. (A2) takes the simple form

$$H_\alpha\left(AH_\alpha{}^2+\sum_{\beta=1}^{M}H_\beta{}^2-B'\right)=0\,,\quad \alpha=1,\ldots,M\,. \tag{A3}$$

For any $\alpha$, we have the choice that $H_\alpha$ either is zero or satisfies the equation

$$AH_\alpha{}^2+\sum_{\beta=1}^{M}H_\beta{}^2-B'=0 \tag{A4}$$

which gives a nonzero solution for $H_\alpha$. The most general solution is then given by

$$AH_\alpha{}^2+\sum_{\beta=1}^{L}H_\beta{}^2-B'=0\,,\quad \alpha=1,\ldots,L \tag{A5}$$

and

$$H_\alpha=0\,,\quad \alpha=L+1,\ldots,M \tag{A6}$$

with $L=0,\ldots,M$. From (A5) we deduce that all the nonzero $H_\alpha{}^2$ are equal and given by

$$H_\alpha{}^2=H^{*2}=\frac{B'}{A+L}\,,\quad \alpha=1,\ldots,L\,. \tag{A7}$$

The next step is to show that in order for the derivative matrix to have all eigenvalues with only nonpositive real parts we must have $L=0$. Namely, $H_\alpha^*(\infty)$ are zero for all $\alpha$—the result asserted in Sec. III.

The derivative matrix is of the form

$$\underline{D}_{\alpha\beta}=\begin{bmatrix} C & D & \cdots & D & & & & & \\ D & C & \cdots & D & & & & & \\ \vdots & \vdots & & \vdots & & & 0 & & \\ D & D & \cdots & C & & & & & \\ & & & & -B' & & & & \\ & & & & & -B' & & & \\ & & & & & & \ddots & & \\ & & 0 & & & & & & \\ & & & & & & & & -B' \end{bmatrix}, \tag{A8}$$

where the submatrix in the upper left-hand corner is $L\times L$ and where

$$C=(3A+L+2)H^{*2}-B'\,,$$

$$D=2H^{*2}\,.$$

We note the $L\times L$ square in the upper left corner in the $\underline{D}$ matrix may be written as

$$(C-D)1+D\underline{T}\,, \tag{A9}$$

where $\underline{T}$ is an $L\times L$ square matrix with *each* element equal to unity, hence satisfying the identity

$$\underline{T}^2=L\underline{T}\,. \tag{A10}$$

From this we conclude that the eigenvalues of $T$ are either zero or $L$. Since the trace is invariant, $\underline{T}$ must be of the form

$$\underline{T}=\begin{bmatrix} L & & & & & & \\ & 0 & & & & & \\ & & 0 & & & & \\ & & & 0 & & & \\ & & & & \ddots & & \\ & & & & & & 0 \end{bmatrix}$$

when diagonalized. Now the requirement that all the eigenvalues of $\underline{D}_{\alpha\beta}$ be negative demands

$$(3A+L)(H^*)^2+2L(H^*)^2-B'<0\,, \tag{A11a}$$

$$(3A+L)(H^*)^2-B'<0\quad (L>1)\,, \tag{A11b}$$

$$-B'<0\,. \tag{A11c}$$

Since $H^{*2}=B'/(A+L)$ and $B'$ must be $>0$, condition (A11a) cannot be satisfied. So the only possible solution is $L=0$ or $H_\alpha^*=0$ for all $\alpha$. Since $H_\alpha^*$ is nothing but a set of linear combinations of $h_{\alpha\beta}^*/g$, we conclude that $h_{\alpha\beta}^*/g\to 0$ as $t\to\infty$.

## APPENDIX B

In this appendix we will try to illustrate the calculations of the renormalization-group equations for the scalar couplings described in Sec. IV. Since the calculations in all the cases are very similar, we outline the steps in one case and state the results for all the other cases.

### 1. One vector and one antisymmetric tensor in O(*N*)

The scalars which belong to the vector representation of O(*N*) are denoted by $\chi_i$ ($i=1,\ldots,N$), and those belonging to the second-rank tensor representation are denoted by $\phi_{ij}$ ($i,j=1,\ldots,N$), with $\phi_{ij}=-\phi_{ji}$. We can write down the most general quartic self-coupling by contracting all the indices to make it invariant under O($N$):

$$-\mathcal{L}_{\rm int}=\tfrac{1}{2}\lambda_1(\phi_{ij}\phi_{ij})^2+\lambda_2(\phi_{ij}\phi_{jk}\phi_{kl}\phi_{li})+\tfrac{1}{8}\lambda_3(\chi_i\chi_i)^2+\tfrac{1}{2}\lambda_4(\chi_k\chi_k)(\phi_{ij}\phi_{ji})+\tfrac{1}{4}\lambda_5(\chi_i\chi_j)(\phi_{ik}\phi_{kj}). \tag{B1}$$

We also need the couplings between gauge particles and the scalars,

$$\mathcal{L}_g=-[\partial_\mu\phi_{ij}-g(A_{\mu il}\phi_{lj}+A_{\mu jl}\phi_{il})]\times[\partial^\mu\phi_{ij}-g(A^\mu_{ik}\phi_{kj}+A^\mu_{jk}\phi_{ik})]-\tfrac{1}{2}(\partial_\mu\chi_i-gA_{\mu il}\chi_l)(\partial^\mu\chi_i-gA_{\mu ik}\chi_k), \tag{B2}$$

where $A_{\mu ij}=-A_{\mu ji}$ are the gauge particles in O($N$). Though the tensor notations $\phi_{ij}, A_{\mu ij}$ are very convenient for constructing invariants under O($N$), they are very clumsy for working out the Feynman rules, because two indices are required to label these fields. To overcome this, we first make a transformation to go over to one-index labeling by writing

$$\phi_{ij}=B^\alpha_{ij}\phi_\alpha,\quad A_{\mu ij}=B^\alpha_{ij}A^\alpha_\mu, \tag{B3}$$

where $B^\alpha\,[\alpha=1,2,\ldots,\tfrac{1}{2}N(N-1)]$ are the complete set of real antisymmetric $N\times N$ matrices with normalization

$$\mathrm{Tr}(B^\alpha B^\beta)=-\tfrac{1}{2}\delta^{\alpha\beta}. \tag{B4}$$

Then the interactions (B1) and (B2) can be rewritten as

$$-\mathcal{L}_{\rm int}=\tfrac{1}{8}\lambda_1(\phi_\alpha\phi_\alpha)^2+\lambda_2\,\mathrm{Tr}(B^\alpha B^\beta B^\gamma B^\delta)\phi^\alpha\phi^\beta\phi^\gamma\phi^\delta+\tfrac{1}{8}\lambda_3(\chi_i\chi_i)^2+\tfrac{1}{4}\lambda_4(\chi_i\chi_i)(\phi_\alpha\phi_\alpha)+\tfrac{1}{4}\lambda_5(\chi_i\chi_j)(\phi^\alpha\phi^\beta)(B^\alpha B^\beta)_{ij} \tag{B5}$$

and

$$\mathcal{L}_g=-\tfrac{1}{2}(\partial_\mu\phi_\alpha-gA^\beta_\mu\theta^\beta_{\alpha\gamma}\phi_\gamma)(\partial^\mu\phi_\alpha-gA^{\rho\mu}\theta^\rho_{\alpha\delta}\phi_\delta)-\tfrac{1}{2}(\partial_\mu\chi_i-gA^\alpha_\mu B^\alpha_{ij}\chi_j)(\partial^\mu\chi_i-gA^{\mu\beta}B^\beta_{ik}\chi_k) \tag{B6}$$

with

$$\theta^\beta_{\alpha\gamma}=2i\,\mathrm{Tr}([B^\beta,B^\alpha]B^\gamma). \tag{B7}$$

With the interactions given above, it is straightforward to work out the Feynman rules to calculate various renormalization constants. The renormalization-group equations are given by

$$\begin{aligned}
\frac{d\lambda_1}{dt}&=\frac{1}{16\pi^2}\{[\tfrac{1}{2}N(N-1)+8]\lambda_1^2+2(2N-1)\lambda_1\lambda_2+6\lambda_2^2+N\lambda_4^2+2\lambda_4\lambda_5-6(N-2)g^2\lambda_1+9g^4\},\\
\frac{d\lambda_2}{dt}&=\frac{1}{16\pi^2}[(2N-1)\lambda_2^2+12\lambda_1\lambda_2+\tfrac{1}{2}\lambda_5^2-6(N-2)g^2\lambda_2+\tfrac{3}{2}(N-8)g^4],\\
\frac{d\lambda_3}{dt}&=\frac{1}{16\pi^2}[(N+8)\lambda_3^2+\tfrac{1}{2}N(N-1)\lambda_4^2+(N-1)\lambda_4\lambda_5+\tfrac{1}{4}(N-1)\lambda_5^2-3(N-1)\lambda_3g^2+\tfrac{3}{4}g^4(N-1)],\\
\frac{d\lambda_4}{dt}&=\frac{1}{16\pi^2}\{[\tfrac{1}{2}N(N-1)+2]\lambda_1\lambda_4+(2N-1)\lambda_2\lambda_4+\tfrac{1}{2}(N-1)\lambda_1\lambda_5+\lambda_2\lambda_5+(N+2)\lambda_3\lambda_4+\lambda_3\lambda_5+2\lambda_4^2\\
&\qquad+\tfrac{1}{2}\lambda_5^2-\tfrac{3}{2}(3N-5)g^2\lambda_4+\tfrac{3}{4}g^4\},\\
\frac{d\lambda_5}{dt}&=\frac{1}{16\pi^2}[2\lambda_1\lambda_5+(N-1)\lambda_2\lambda_5+2\lambda_3\lambda_5+4\lambda_4\lambda_5+\tfrac{1}{2}N\lambda_5^2-\tfrac{3}{2}(3N-5)g^2\lambda_5+\tfrac{3}{4}(N-4)g^4].
\end{aligned} \tag{B8}$$

The calculations of these coefficients in the above equations are straightforward but tedious. For example, consider the diagrams given by Fig. 4, where both external lines and internal lines belong to antisymmetric tensor $\phi_\alpha$ with vertices corresponding to $\lambda_2$. These diagrams contain the internal symmetry factors

$$\begin{aligned}
S_a&=\sum_{\rho\xi}\mathrm{Tr}(B^\alpha B^\beta B^\rho B^\xi)_s\,\mathrm{Tr}(B^\gamma B^\delta B^\rho B^\xi)_s,\\
S_b&=\sum_{\rho\xi}\mathrm{Tr}(B^\alpha B^\gamma B^\rho B^\xi)_s\,\mathrm{Tr}(B^\beta B^\delta B^\rho B^\xi)_s,\\
S_c&=\sum_{\rho\xi}\mathrm{Tr}(B^\alpha B^\delta B^\rho B^\xi)_s\,\mathrm{Tr}(B^\beta B^\gamma B^\rho B^\xi)_s,
\end{aligned} \tag{B9}$$

where $\mathrm{Tr}(ABCD)_s$ denotes the totally symmetric combination of $A$, $B$, $C$, and $D$. We need to sum over the intermediate states $\rho$, $\xi$ and project out the term $\mathrm{Tr}(B^\alpha B^\beta B^\gamma B^\delta)_s$ in order to calculate their contribution to $d\lambda_2/dt$. This can be done by the observation that $B^\alpha\,[\alpha=1,2,\ldots,\tfrac{1}{2}N(N-1)]$ form the complete set of real antisymmetric $N\times N$ matrices. We can use them to expand any arbitrary real antisymmetric $N\times N$ matrix $M$ in the form

$$M_{ij}=\sum_\alpha B^\alpha_{ij}a^\alpha,\quad M_{ij}=-M_{ji}. \tag{B10}$$

The coefficient $a^\alpha$ can be calculated from Eq. (B4):

$$a^\alpha=-2(M_{ij}B^\alpha_{ji})=-2(\mathrm{Tr}MB^\alpha). \tag{B11}$$

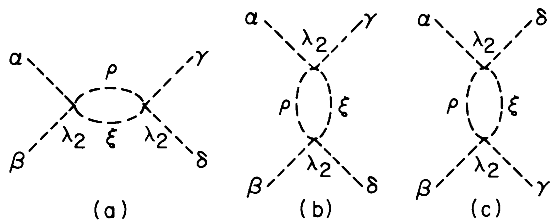


FIG. 4. The three graphs contributing to the $\lambda_2^2$ terms in the renormalization-group equations for $d\lambda_2/dt$, in the case where the scalar fields belong to the antisymmetric-tensor-plus-one-vector representation of O($N$).

Substituting back into Eq. (B10), we get

$$M_{ij}=-2\sum_{\alpha}(B^{\alpha}_{ij}B^{\alpha}_{lk})M_{kl}\,,\quad i,j=1,\ldots,N\,. \tag{B12}$$

Taking into account the antisymmetric nature of $M$, we can work out the completeness relation,

$$\sum_{\alpha}B^{\alpha}_{ij}B^{\alpha}_{lk}=-\tfrac{1}{4}(\delta_{ik}\delta_{jl}-\delta_{jk}\delta_{il})\,. \tag{B13}$$

With this relation, the summation in (B9) can be worked out to give the coefficient of $\lambda_2^2$ in the equation for $d\lambda_2/dt$. It is clear from this example how the calculations are carried out. The case of an antisymmetric representation of O($N$) can be obtained from this example by setting all the coupling constants containing scalars $\lambda_3, \lambda_4, \lambda_5$ equal to zero. Now we give the results in all the other cases.

### 2. Symmetric second-rank tensor representation in O($N$)

The couplings are

$$-\mathcal{L}_{\rm int}=\tfrac{1}{2}\lambda_1(\phi_{ij}\phi_{ij})^2+\lambda_2(\phi_{ij}\phi_{jk}\phi_{kl}\phi_{li})\,,$$

with $\phi_{ij}=\phi_{ji}$ and $\sum_i\phi_{ii}=0$, and

$$-\mathcal{L}_g=-[\partial_\mu\phi_{ij}-g(A_{\mu il}\phi_{lj}+A_{\mu jl}\phi_{il})]\times[\partial^\mu\phi_{ij}-g(A^\mu_{ik}\phi_{kj}+A^\mu_{jk}\phi_{ik})]\,.$$

The stability equations are given by

$$\frac{d\lambda_1}{dt}=\frac{1}{16\pi^2}\left\{\left[\frac{N(N+1)}{2}+7\right]\lambda_1^2+\frac{2(2N^2+3N-6)}{N}\lambda_1\lambda_2+\frac{6(N^2+6)}{N^2}\lambda_2^2-6N\lambda_1g^2+9g^4\right\},$$

$$\frac{d\lambda_2}{dt}=\frac{1}{16\pi^2}\left\{12\lambda_1\lambda_2+\frac{2N^2+9N-36}{N}\lambda_2^2-6N\lambda_2g^2+\tfrac{3}{2}Ng^4\right\}.$$

### 3. $m$ vector representations in O($N$)

We label these vector representations by $\vec\phi_1, \vec\phi_2, \ldots, \vec\phi_m$. The couplings are given by

$$-\mathcal{L}_{\rm int}=\sum_{i=1}^{m}\tfrac{1}{8}\lambda_i(\vec\phi_i\cdot\vec\phi_i)^2+\sum_{i\neq j}^{m}\tfrac{1}{8}\rho_{ij}(\vec\phi_i\cdot\vec\phi_i)(\vec\phi_j\cdot\vec\phi_j)+\sum_{i\neq j}^{m}\tfrac{1}{4}\eta_{ij}(\vec\phi_i\cdot\vec\phi_j)(\vec\phi_i\cdot\vec\phi_j)$$

and

$$\mathcal{L}_g=\sum_{i=1}^{m}(\partial_\mu\phi_a^{(i)}-gA_{\mu ab}\phi_b^{(i)})(\partial_\mu\phi_a^{(i)}-gA_{\mu ac}\phi_c^{(i)})\,.$$

We have assumed that $\mathcal{L}_{\rm int}$ is invariant under any reflection $\vec\phi_i\to-\vec\phi_i$ to make the thing as simple as possible. If we further impose the symmetry that $\mathcal{L}_{\rm int}$ is invariant under any interchange among $\vec\phi_1, \ldots, \vec\phi_m$, we can get the stability equations in the form

$$\frac{d\lambda}{dt}=\frac{1}{16\pi^2}[(N+8)\lambda^2+(m-1)N\rho^2+4(m-1)\rho\eta+4(m-1)\eta^2-3(N-1)g^2\lambda+\tfrac{3}{4}g^4(N-1)],$$

$$\frac{d\rho}{dt}=\frac{1}{16\pi^2}[2(N+2)\rho\lambda+4\eta\lambda+N(m-2)\rho^2+2\rho^2+2\eta^2+4(m-2)\eta\rho-3(N-1)g^2\rho+\tfrac{3}{4}g^4],$$

$$\frac{d\eta}{dt}=\frac{1}{16\pi^2}[4\lambda\eta+2(m-2)\eta^2+4\eta\rho+(N+2)\eta^2-3(N-1)g^2\eta+\tfrac{3}{8}(N-2)g^4],$$

where $\lambda_i=\lambda$ $(i=1,\ldots,m)$, $\rho_{ij}=\rho$ $(i,j=1,\ldots,m)$, and $\eta_{ij}=\eta$ $(i,j=1,\ldots,m)$.

### 4. Vector representation in SU($N$)

We denote the complex vector representation by $\psi_i$ $(i=1,2,\ldots,n)$, and its conjugate by $\psi^i=(\psi_i)^*$ $(i=1,\ldots,N)$. The gauge particles are denoted by $A^j_{\mu i}$ $(i,j=1,\ldots,n)$, with $A^j_{\mu i}=(A^i_{\mu j})^*$ and $\sum_i A^i_{\mu i}=0$. Just as in O($N$), the SU($N$)-invariant quartic and gauge couplings can be written down by contracting all the indices:

$$-\mathcal{L}_{\rm int}=\tfrac{1}{2}\lambda(\psi^i\psi_i)^2\,,$$

$$\mathcal{L}_g=(\partial_\mu\psi^i+igA^i_{\mu j}\psi^j)(\partial^\mu\psi_i-igA^k_{\mu i}\psi_k)\,.$$

The stability equation is given by

$$\frac{d\lambda}{dt}=\frac{1}{8\pi^2}\left[(N+4)\lambda^2-\frac{3(N^2-1)}{N}\lambda g^2+\frac{3(N-1)(N^2+2N-2)}{4N^2}g^4\right].$$

### 5. Two vector representations in SU($N$)

We denote these two vector representations by $\psi_i$ and $\psi_i'$. The quartic and gauge couplings are

$$-\mathcal{L}_{\rm int}=\tfrac{1}{2}\lambda_1(\psi^i\psi_i)^2+\tfrac{1}{2}\lambda_2(\psi'^i\psi_i')^2+\lambda_3(\psi^i\psi_i)(\psi'^j\psi_j')+\lambda_4(\psi'^i\psi_i)(\psi^j\psi_j')+\tfrac{1}{2}\lambda_5[(\psi'^i\psi_i)^2+(\psi^i\psi_i')^2],$$

$$\mathcal{L}_g=(\partial_\mu\psi_i+igA^j_{\mu i}\psi_j)(\partial^\mu\psi^i-igA^{\mu i}_k\psi^k)+(\partial_\mu\psi_i'+igA^j_{\mu i}\psi_j')(\partial^\mu\psi'^i-igA^{\mu i}_k\psi'^k),$$

where we have assumed that $\mathcal{L}_{\rm int}$ has the symmetry $\psi^i\to-\psi^i$.

The stability equations are given by

$$\frac{d\lambda_1}{dt}=\frac{1}{8\pi^2}\left[(N+4)\lambda_1^2+N\lambda_3^2+2\lambda_3\lambda_4+\lambda_4^2+\lambda_5^2-\frac{3(N^2-1)}{N}\lambda_1g^2+\frac{3(N-1)(N^2+2N-2)}{4N^2}g^4\right],$$

$$\frac{d\lambda_2}{dt}=\frac{1}{8\pi^2}\left[(N+4)\lambda_2^2+N\lambda_3^2+2\lambda_3\lambda_4+\lambda_4^2+\lambda_5^2-\frac{3(N^2-1)}{N}\lambda_2g^2+\frac{3(N-1)(N^2+2N-2)}{4N^2}g^4\right],$$

$$\frac{d\lambda_3}{dt}=\frac{1}{8\pi^2}\left\{[(N+1)\lambda_3+\lambda_4](\lambda_1+\lambda_2)+2\lambda_3^2+\lambda_4^2+\lambda_5^2-\frac{3(N^2-1)}{N}\lambda_3g^2+\frac{3(N^2+2)}{4N^2}g^4\right\},$$

$$\frac{d\lambda_4}{dt}=\frac{1}{8\pi^2}\left[\lambda_4(\lambda_1+\lambda_2)+4\lambda_3\lambda_4+N\lambda_4^2+(N+2)\lambda_5^2-\frac{3(N^2-1)}{N}\lambda_4g^2+\frac{3(N^2-4)}{4N}g^4\right],$$

$$\frac{d\lambda_5}{dt}=\frac{1}{8\pi^2}\left\{\lambda_5[(\lambda_1+\lambda_2)+4\lambda_3+2(N+1)\lambda_4]-\frac{3(N^2-1)}{N}\lambda_5g^2\right\}.$$

### 6. $m$ vector representations in SU($N$)

These $m$ sets of vector representations are denoted by $\psi^{(1)},\psi^{(2)},\ldots,\psi^{(m)}$.

The quartic and gauge couplings are

$$-\mathcal{L}_{\rm int}=\sum_{a=1}^{m}\tfrac{1}{2}\lambda_a(\psi^{(a)i}\psi_i^{(a)})^2+\sum_{a\neq b}^{m}\tfrac{1}{2}\eta_{ab}(\psi^{(a)i}\psi_i^{(a)})(\psi^{(b)j}\psi_j^{(b)})+\sum_{a\neq b}^{m}\tfrac{1}{2}\rho_{ab}(\psi^{(a)i}\psi_i^{(b)})(\psi^{(b)j}\psi_j^{(a)})+\sum_{a\neq b}\tfrac{1}{4}\sigma_{ab}[(\psi^{(a)i}\psi_i^{(b)})^2+(\psi^{(b)i}\psi_i^{(a)})^2]$$

and

$$\mathcal{L}_g=\sum_{a=1}^{m}(\partial_\mu\psi^{(a)i}+igA^i_{\mu j}\psi^{(a)j})(\partial^\mu\psi^{(a)i}-igA^{\mu i}_k\psi^{(a)k}),$$

where we have assumed the symmetry under any reflection $\psi_i^{(a)}\to-\psi_i^{(a)}$. Again, if we further assume the interchange symmetries among these $m$ vectors, the stability equations become

$$\frac{d\lambda}{dt}=\frac{1}{8\pi^2}\left[(N+4)\lambda^2+(m-1)(N\eta^2+\rho^2+\sigma^2)+2(m-1)\eta\rho-\frac{3(N^2-1)}{N}\lambda g^2+\frac{3(N-1)(N^2+2N-2)}{4N^2}g^4\right],$$

$$\frac{d\eta}{dt}=\frac{1}{8\pi^2}\left[2(N+1)\lambda\eta+2\lambda\rho+\rho^2+\sigma^2+2\eta^2+N(m-2)\eta^2+2(m-2)\eta\rho-\frac{3(N^2-1)}{N}\eta g^2+\frac{3(N^2+2)}{4N^2}g^4\right],$$

$$\frac{d\rho}{dt}=\frac{1}{8\pi^2}\left[2\lambda\rho+N\rho^2+(N+2)\sigma^2+4\eta\rho+(m-2)\rho^2-\frac{3(N^2-1)}{N}\rho g^2+\frac{3(N^2-4)}{4N}g^4\right],$$

$$\frac{d\sigma}{dt}=\frac{1}{8\pi^2}\left\{\sigma\left[2\lambda+4\eta+2(N+1)\rho-\frac{3(N^2-1)}{N}g^2+(m-2)\sigma\right]\right\}.$$

### 7. One vector and one adjoint representations in SU($N$)

We denote the vector representation by $\chi_i$ $(i=1,\ldots,N)$ and the adjoint representation by $\psi_i^j$ $(i,j=1,\ldots,N)$, with $\psi_i^j=(\psi_j^i)^*$ and $\sum_i\psi_i^i=0$. The quartic and gauge couplings are given by

$$-\mathcal{L}_{\rm int}=\tfrac{1}{2}\lambda_1(\psi_i^j\psi_j^i)^2+\lambda_2(\psi_i^j\psi_j^k\psi_k^l\psi_l^i)+\tfrac{1}{2}\lambda_3(\chi^i\chi_i)^2+\tfrac{1}{4}\lambda_4(\chi^i\chi_i)(\psi_j^k\psi_k^j)+\lambda_5(\chi_i\chi^j)(\psi_k^i\psi_j^k),$$

$$\mathcal{L}_g=(\partial_\mu\chi^i+igA^i_{\mu j}\chi^j)(\partial^\mu\chi_i-igA^k_{\mu i}\chi_k)+(\partial_\mu\psi_j^i+igA^i_{\mu k}\psi_j^k-igA^k_{\mu j}\psi_k^i)(\partial^\mu\psi_i^j-igA^{\mu j}_l\psi_i^l+igA^{\mu l}_i\psi_l^j).$$

The stability equations are

$$\frac{d\lambda_1}{dt}=\frac{1}{8\pi^2}\left[\frac{N^2+7}{2}\lambda_1^{\,2}+\frac{2(2N^2-3)}{N}\lambda_1\lambda_2+\frac{6(N^2+3)}{N^2}\lambda_2^{\,2}+N\lambda_4^{\,2}+2\lambda_4\lambda_5-6N\lambda_1 g^2+9g^4\right],$$

$$\frac{d\lambda_2}{dt}=\frac{1}{8\pi^2}\left[\frac{2(N^2-9)}{N}\lambda_2^{\,2}+6\lambda_1\lambda_2+\tfrac{1}{2}\lambda_5^{\,2}-6N\lambda_2 g^2+\frac{3N}{2}g^4\right],$$

$$\frac{d\lambda_3}{dt}=\frac{1}{8\pi^2}\left[(N+4)\lambda_3^{\,2}+\frac{N^2-1}{2}\lambda_4^{\,2}+\frac{N^2-1}{N}\lambda_4\lambda_5+\lambda_5^{\,2}\frac{(N-1)(N^2+2N-2)}{4N^2}-\frac{3(N^2-1)}{N}\lambda_3 g^2+\frac{3(N-1)(N^2+2N-2)}{4N^2}g^4\right],$$

$$\frac{d\lambda_4}{dt}=\frac{1}{8\pi^2}\left[\frac{N^2+1}{2}\lambda_1\lambda_4+\frac{2N^2-3}{N}\lambda_2\lambda_4+(N+1)\lambda_3\lambda_4+\frac{N^2-1}{2N}\lambda_1\lambda_5+\frac{N^2+3}{N^2}\lambda_2\lambda_5\right.$$
$$\left.+\lambda_3\lambda_5+2\lambda_4^{\,2}+\tfrac{1}{2}\lambda_5^{\,2}-\frac{3(3N^2-1)}{2N}\lambda_4 g^2+\tfrac{3}{4}g^4\right],$$

$$\frac{d\lambda_5}{dt}=\frac{1}{8\pi^2}\left\{\lambda_5\left[\lambda_1+\frac{N^2-6}{N}\lambda_2+\lambda_3+2\lambda_4+\frac{N^2-4}{2N}\lambda_5-\frac{3(3N^2-1)}{2N}g^2\right]+\tfrac{3}{4}Ng^4\right\}.$$

### 8. Antisymmetric tensor in SU($N$)

We consider the antisymmetric tensor representation of SU($N$) for completeness. (The symmetric tensor representation was covered by Ref. 13.) We denote the antisymmetric tensor representation by $\psi_{ij}$ $(i,j=1,\ldots,n)$, with $\psi_{ij}=-\psi_{ji}$. Also, the conjugate fields $\psi^\dagger_{ij}$ are denoted $\psi^{ij}$. The quartic and gauge couplings are given by

$$-\mathcal{L}_{\rm int}=\tfrac{1}{2}\lambda(\psi^{ij}\psi_{ij})^2+\tfrac{1}{2}\eta\psi^{ij}\psi_{jk}\psi^{kl}\psi_{li},$$
$$\mathcal{L}_g=(\partial_\mu\psi^{ij}+igA^{\,i}_{\mu l}\psi^{lj}+igA^{\,j}_{\mu l}\psi^{il})$$
$$\times(\partial^\mu\psi_{ji}-igA_j^{\mu l}\psi_{li}-igA_i^{\mu l}\psi_{jl}).$$

The stability equations are

$$\frac{d\lambda}{dt}=\frac{1}{8\pi^2}\left[\frac{N(N-1)+6}{2}\lambda^2+\frac{N-1}{2}\eta\lambda+\frac{\eta^2}{16}\right.$$
$$\left.-\frac{6(N+1)(N-2)}{N}\lambda g^2+\frac{3}{N^2}(3N^2+8)g^4\right],$$

$$\frac{d\eta}{dt}=\frac{1}{8\pi^2}\left[4\lambda\eta+\frac{N-2}{4}\eta^2-\frac{6(N+1)(N-2)}{N}\eta g^2\right.$$
$$\left.+\frac{12(N^2-4N-16)}{N}g^4\right].$$

### 9. Contributions of various representations to $S_3(R)$

Using the trick of introducing the transformation matrix, as described above, we can calculate easily the quantity $S_3(R)$, which is the contribution of scalars to $\beta(g)$ of the gauge coupling constant. We tabulate the results and show them in Table III.

## APPENDIX C

We shall illustrate, in a simple case, the pattern of symmetry breaking via the Higgs mechanism. We refer the reader to Ref. 19 for the details. One of the purposes of this exercise is to emphasize that the form which vacuum expectation values of the scalars can take in a given theory is dictated by the structure of the Lagrangian and cannot be set to some arbitrary values—a point which is often overlooked in the various attempts to construct gauge models of weak and electromagnetic interactions. Let us consider a gauge theory based on the scalars belonging to the vector representation of O($N$). The corresponding classical potential $V(\phi)$ is given by

$$V(\phi)=-\tfrac{1}{2}\mu^2\phi_i\phi_i+\tfrac{1}{4}\lambda(\phi_i\phi_i)^2,\quad \lambda>0 \tag{C1}$$

TABLE III. Values of $S_3(R)$ of various groups and representations.

| Representation | $S_3(R)$ for O($N$) | $S_3(R)$ for SU($N$) |
|---|---|---|
| Vector | $\frac{1}{2}$ | $\frac{1}{2}$ |
| Adjoint | $\frac{1}{2}(N-2)$ | $N$ |
| Antisymmetric 2nd-rank tensor | $\frac{1}{2}(N-2)$ | $\frac{1}{2}(N-2)$ |
| Symmetric 2nd-rank tensor | $\frac{1}{2}(N+2)$ | $\frac{1}{2}(N+2)$ |
| Totally symmetric 3rd-rank tensor | $\frac{1}{4}(N+1)(N+4)$ | $\frac{1}{4}(N+2)(N+3)$ |
| Totally symmetric $k$th-rank tensor | $\frac{(N+1)(N+2)\cdots(N+k-2)}{2(k-1)!}(N+2k-2)$ | $\frac{(N+2)(N+3)\cdots(N+k)}{2(k-1)!}$ |

and the coupling between gauge particles and scalars is given by

$$\mathcal{L}_g = \tfrac{1}{2}(\partial_\mu \phi_i - gA_{\mu ij}\phi_j)(\partial^\mu \phi_i - gA^\mu_{ik}\phi_k), \qquad \text{(C2)}$$

where $A_{\mu ij} = -A_{\mu ji}$ $(i, j = 1, \ldots, n)$ are the gauge fields in O($N$). The vacuum expectation value of $\phi_i$, $\langle\phi_i\rangle$, is chosen to be that which gives the absolute minimum for the potential $V(\phi)$. To get this minimum we can calculate its first derivatives,

$$\frac{\partial V}{\partial \phi_i} = (-\mu^2 + \lambda\phi_j\phi_j)\phi_i = 0, \quad i = 1, 2, \ldots, N. \qquad \text{(C3)}$$

The solution corresponding to the spontaneously broken symmetry is given by

$$|\vec{\phi}|^2 = \phi_j\phi_j = \mu^2/\lambda. \qquad \text{(C4)}$$

Notice that the requirement of the minimum only determines the length of the vector $\vec{\phi}$. This is because the potential only depends on the length of the vector $\vec{\phi}$. We can choose it to be of the form

$$\vec{\phi} = (0, 0, \ldots, 0, \mu^2/\lambda). \qquad \text{(C5)}$$

All the other solutions are equivalent to this one because they can be reached by O($N$) rotation. The gauge particles obtain their masses through the coupling given in (C2),

$$-\tfrac{1}{2}g^2 A_{\mu ij}\langle\phi_j\rangle A^\mu_{ik}\langle\phi_k\rangle.$$

With the solution given in (C5), it is clear that $A_{\mu in}$ $(i = 1, \ldots, N-1)$ become massive, while $A_{\mu ij}$ $(i, j = 1, \ldots, N-1)$ remain massless, corresponding to the generators of O($N-1$). The symmetry is broken from O($N$) to O($N-1$). An easy way to understand this result is the observation that the vacuum expectation value $\langle\phi_i\rangle$ given in (C5) is invariant under all the rotations leaving the $N$ axis unchanged, which is the subgroup O($N-1$). With this picture in mind, it is very easy to get the results in the cases where there are more than one vector representations. Consider the case with two such vector representations, say $\vec{\phi}_1$ and $\vec{\phi}_2$. The O($N$)-invariant potential can depend only on the length of each vector and the angle between them, $|\vec{\phi}_1|$, $|\vec{\phi}_2|$, and $|\vec{\phi}_1 \cdot \vec{\phi}_2|$. The solutions for the minimum fix up the magnitudes for these three variables. We can choose the first vector with only the first component nonzero, and the second vector with the first two components nonzero in order to satisfy these conditions. The symmetry is then reduced from O($N$) to O($N-2$). We can generalize this argument to any number of vector representations with the results that O($N$) → O($N-m$) for $m$ sets of vector representations. In particular, it takes ($N-1$) sets of vectors to break the O($N$) symmetry completely. Exactly the same argument can be applied to the SU($N$) group.

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# Classical direct interstring action*

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We generalize the classical action-at-a-distance theory between point particles to include one-dimensionally extended objects (strings) in space-time. We build parametrization-invariant couplings which lead to equations of motion for strings in each others' influence. The direct coupling of the area elements of the world sheets of the strings is considered in detail, from which we define an antisymmetric adjunct field. We find that, for a given interaction, the nature of the forces depends on the type of strings involved, that is, open- vs closed-ended. Our coupling can be understood in terms of states appearing in the Veneziano and Shapiro-Virasoro models in 26 dimensions. However, we find an additional massive pseudovector field which arises from the interaction between the "Reggeon" and "Pomeron" sectors of this dual model.

## I. INTRODUCTION

The dual resonance models,[1] whose aim is a self-contained description of the strong interactions, have of late been understood in terms of a strikingly simple and beautiful picture. On the one hand, the states of motion of a one-dimensionally extended object (string)[2] with open ends are identified with the mesonic resonances which mediate the strong interactions,[3] while the "background" (Pomeron) is to be related to the states of motion of strings which close on themselves. On the other hand, Mandelstam[4] has shown that the Veneziano amplitudes can be obtained by breaking and joining open strings, thus completing the description. It is therefore rather unfortunate that such conceptual simplicity is spoiled by the presence of tachyons, and long-range forces, all in a 26-dimensional space-time.[5] Still, these problems appear only in the quantization procedure, and do not subtract from the appeal of the classical description. A difficulty in overcoming these defects is that the strings have so far been described in terms of their world sheets rather than by the fields associated with them. One may hope therefore that the development of a more powerful formalism might alleviate and perhaps solve the aforesaid problems.

Nevertheless, at the classical level this remains a very beautiful theory which does not make use of a field description. In this light, it seems natural to try to understand Mandelstam's interaction as being generated by direct interstring forces. One already knows that Maxwell's theory can be described in terms of such forces, as shown by Feynman and Wheeler.[6] It is our aim in this paper to generalize action-at-a-distance theories to include direct interstring interactions. As a first step, we limit ourselves to a specific type of interaction obtained by analogy to their work. Thus we concern ourselves, in what follows, with a tiny subset of all the possible direct interstring inter-