How Heavy are the Quarks?

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We consider a simple model of mesons as Coulombic bound states of quark-antiquark pairs confined in a box. A fit to the meson spectra and decay rates suggests that $m_u = m_d \approx 0.3$ GeV, $m_s \approx 0.4$ GeV, and $m_c \approx 1.4$ GeV. We also predict new meson systematics and metastable charmed baryons.

A number of attractive arguments have been advanced that the strong interaction is described by the following Lagrangian:

$$\mathcal{L} = -\tfrac{1}{4} G_{\mu\nu}G^{\mu\nu} + \bar{q}(i\not{D} + M)q\,. \tag{1}$$

$\mathcal{L}$ is exactly invariant under the gauge group $SU(3)_{\rm color}$ ($G_{\mu\nu}$ being the gluons) and it is approximately symmetric under $SU(4)_L \otimes SU(4)_R$. This last (global) symmetry is broken by the quark-mass term

$$\bar{q}Mq = c_0 u_0 + c_8 u_8 + c_{15} u_{15}\,, \tag{2}$$

with $u_0 = \bar{q}\lambda_0 q$, etc. Well-known $SU(3)_L \otimes SU(3)_R$ chiral-symmetry arguments applied to the octet pseudoscalar mesons give[1] $m_s \approx 25 m_{\mathfrak{N}}$ (u and d quarks shall be collectively called $\mathfrak{N}$). This suggests that $SU(2)_L \otimes SU(2)_R$ is a much better symmetry than SU(3). However, one can question the soundness of the original derivation involving the application of soft-meson techniques to the K's and η. In this paper we shall argue that q_s and $q_{\mathfrak{N}}$ in fact have comparable masses.

An important feature of the above Lagrangian is that the effective coupling strength becomes weak at short distances ("asymptotic freedom")[2]; and thus to the first order it has the structure of an Abelian theory. This unique feature of the non-Abelian gauge theory led Appelquist and Politzer[3] and De Rújula and Glashow[4] to suggest that the $\psi(3105)$ and $\psi'(3695)$ are Coulombic bound states of charmed quark and antiquark pairs: (ortho) charmonium.

Asymptotic freedom also suggests, very plausibly, that the forces grow without bound between receding colored particles ("color confinement"). In this paper we shall present a very simple model of the mesons as quark-antiquark pairs bound by the potential[5]

$$V(r) = \begin{cases} -\bar{g}^2/4\pi r & \text{for } r < R\,, \\ \infty & \text{for } r > R\,. \end{cases} \tag{3}$$

In short, what we have is a "hydrogen atom in a box." The confinement effects are crudely incorporated by an infinitely high square well. As we shall see, even this simple model yields a reasonable picture of the meson spectra and decay rates. In Eq. (3), $\bar{g}$ is the strong-interaction effective coupling constant. To the lowest order, the coupling strengths $\alpha_s(m) = \bar{g}^2/4\pi$ at different mass ranges are related according to

$$\alpha_s(m) = \alpha_s(m_0)\left[1 + \frac{25}{12\pi}\alpha_s(m_0)\ln\left(\frac{m^2}{m_0^2}\right)\right]^{-1}. \tag{4}$$

The calculation itself is straightforward: It involves locating the zeros of the Coulomb wave function (a confluent hypergeometric function) at $r = R$. For the present time we have done this numerically, and it is hoped that we shall be able to find an analytic expression for the locations of these zeros in the future. Although the actual fitting of the meson systematics with our model did not proceed completely in a "linear fashion," the basic logical steps with which the parameters are determined are as follows:

(i) The scale of the effective couplings is fixed at the outset by the ratio of the hadronic and leptonic decay widths of $\psi(3105)$. Anticipating that the model should work best for the $(\bar{c}c)$ system and that the $\psi(3105)$, the 1^3S state, should have a mass close to twice the charmed-quark mass, we have[3] ($\alpha \equiv \frac{1}{137}$)

$$\alpha_s^{\,3} = \frac{\pi\alpha^2}{\pi^2 - 9}\,\frac{18}{5}\,\frac{\Gamma(\psi \to \text{hadrons})}{\Gamma(\psi \to l^+ l^-)}\,. \tag{5}$$

This yields $\alpha_s(3\text{ GeV}) = \frac{1}{4}$ and leads to $\alpha_s(1\text{ GeV}) = 0.4$, $\alpha_s(0.7\text{ GeV}) = 0.49$, etc., indicating that the model may be applicable down to mesons in the 0.5-GeV range.

(ii) In order to calculate the absolute decay rate of $V \to l^+ l^-$ we need the wave functions at the origin, $f(0)$:

$$\Gamma(V \to l^+ l^-) = (16\pi\alpha^2 C_V^{\,2}/M_V^{\,2})|f(0)|^2\,, \tag{6}$$

with $C_\rho = 1/\sqrt{2}$, $C_\omega = 1/3\sqrt{2}$, $C_\varphi = \frac{1}{3}$, and $C_\psi = \frac{2}{3}$.

It turns out that the $f(0)$'s are most sensitive to the confinement radius. For the 1S states of the $\bar{q}q$ systems the sizes of the square wells are constrained to be

$$R_{(\bar{c}c)} = 0.82 \text{ fm}, \quad R_{(\bar{s}s)} = 1.6 \text{ fm}, \quad R_{(\bar{\mathfrak{N}}\mathfrak{N})} = 2.0 \text{ fm}, \tag{7}$$

which seem to us to be completely reasonable.

(iii) The remaining parameters, the quark masses, are then determined by the masses of the 1S states. To be sure there is the question of hyperfine splitting between ^3S and ^1S states, but we found that the range of allowed values of quark masses is severely restricted. Our fit to the meson spectrum yields

$$m_{\mathfrak{N}} = 275 \text{ MeV}, \quad m_s = 400 \text{ MeV}, \quad m_c = 1400 \text{ MeV}. \tag{8}$$

This is our main conclusion: If such a model has any physical significance, for mesons above 0.5 GeV, then it is difficult to have quarks differing greatly in mass from the above values.

Given the quark masses we can immediately convert them to c's in Eq. (2):

$$c_0 = 830 \text{ MeV}, \quad c_8 = -72 \text{ MeV},$$

$$c_{15} = -663 \text{ MeV},$$

and

$$x \equiv (m_c - m_{\mathfrak{N}})/(m_s - m_{\mathfrak{N}}) = 9. \tag{9}$$

It would seem to us that if the dynamical approach to the mesons as systems of light quarks moving nonrelativistically in a confined "bag" is at all correct, then linear (instead of quadratic) mass formulas will be physically relevant, since the mass is the linear sum of quark masses and an interaction energy term. For the extremely low-lying π's [and possibly K's] the above picture clearly breaks down with this argument. The Lagrangian in (1) and (2) gives us the following mass relation[5]:

$$\tfrac{1}{2}(M_\psi - M_\rho) = x(M_{K^*} - M_\rho); \tag{10}$$

with our own value of $x = 9$ [Eq. (9)], the agreement is excellent (1168 MeV on left-hand side and 1098 MeV on right-hand side).

Encouraged by the success of the above mass formulas and with $x = 9$ we can calculate all the charmed-baryon masses through the standard relations.[6] In particular, we have

$$M_{C_0} - M_N = M_A - \tfrac{1}{4}(3M_\Sigma - M_\Lambda) = x(M_\Lambda - M_N), \tag{11}$$

$$M_{X_{\mathfrak{N}}} - M_N = M_{X_s} - M_\Sigma = x(M_\Xi - M_N), \tag{12}$$

where C_0^+, $A^{+,0}$, X_s, and $X_{\mathfrak{N}}$ are, respectively, spin-$\frac{1}{2}^+$ baryons having quantum numbers (I, S, C) equal to $(0, 0, 1)$, $(\frac{1}{2}, -1, 1)$, $(0, -1, +2)$, and $(\frac{1}{2}, 0, 2)$. Equations (11) and (12) yield

$$M_{C_0^+} = 2532 \text{ MeV}, \quad M_{A^{+,0}} = 2769 \text{ MeV}, \quad M_{X_{\mathfrak{N}}} = 4341 \text{ MeV}, \quad M_{X_s} = 4595 \text{ MeV}. \tag{13}$$

The strong decay processes are

$$X_{\mathfrak{N}} \to C_0 + D \;\; (\hookrightarrow N + D), \qquad X_s \to A + D \;\; (\hookrightarrow \Lambda + D),$$

if the charmed meson D ($c\bar{\mathfrak{N}}$) has mass less than 1.6 GeV. Such a low D mass is clearly ruled out by experiments, since $\psi'(3695)$ has a narrow width at 3.7 GeV. As we shall see our model predicts a D-meson mass of about 2 GeV. Consequently these charmed baryons will be metastable and decay weakly with a lifetime on the order of 10^{-11} sec. Although this prediction is sensitive to the value of x, even with x as large as 11 these baryons will be stable for our D mass; and, in any case, C_0^+ will probably be stable.

Returning to the meson model, with all parameters fixed as above, the $\varphi \to 3\pi$ decay rate is given by

$$\Gamma(\varphi \to 3\pi) = \tfrac{5}{18}[\tfrac{16}{9}(\pi^2 - 9)\,\alpha_s^3(1 \text{ GeV})/m_s^2]\,|f(0)|^2. \tag{14}$$

Numerical computation yields $|f(0)|^2 \approx 0.005$ and a width of 0.86 MeV, in agreement with experiment.

We next calculate the spectrum of mesons in the $\bar{c}s$, $\bar{c}\mathfrak{N}$, and $\bar{s}\mathfrak{N}$ systems as well as the first few excited levels in the $\bar{\mathfrak{N}}\mathfrak{N}$, $\bar{c}c$, and $\bar{s}s$ systems. Since our fit to the 1S levels in the unmixed systems indicated that R, the confinement radius, is dependent upon the quark masses, an empirical relation which gives good agreement for the unmixed $\bar{q}q$ systems was used to predict R for the mixed systems: The product of R^2 with the $\bar{q}q$ reduced mass is very nearly constant (we do not speculate at this point on the physics of this relation). We expect that, in analogy with diatomic molecular systems, the confinement bag will be stretched for rotationally excited states. Indeed such an effect is noted when the first P-wave states in $\bar{\mathfrak{N}}\mathfrak{N}$ and $\bar{s}s$ are identified with "mean multiplet masses" in the observed meson spectrum. Since in our model the $\psi'(3695)$ is best identified with a 1^3D_1 state, the "stretching" can be determined for $L=2$ excitation. The stretching effects are well represented by an empirical rule

$$\Delta R = \tfrac{1}{30} L(L+1) \text{ fm}; \tag{15}$$

this relation is adopted in all of our subsequent calculations.

Since the masses of the $\bar{q}q$ meson states spread out over a fairly large range, changes in α are significant between system levels. Consequently a sort of iterative "bootstrap" procedure was adopted to calculate the masses of the meson levels. The results of our calculations for the first few levels in each system are collected in Table I. We emphasize that the results for the meson spectrum as well as the leptonic decay rates of ω, ρ, φ, and ψ, the $\varphi \to 3\pi$ decay rate, and the hadronic decay rate of ψ, are all fixed by six parameters: two for the square-well radius, the three quark masses, and the coupling at 3 GeV. We have not included relativistic corrections or spin-orbit effects.

The following aspects of the spectrum in Table I are of particular interest:

(a) $\epsilon(600)$, $I^G(J^P)C = 0^+(0^+)+$, is *not* a bonafide meson state in this model.

(b) The lowest charmed-meson states are predicted to have masses of 1.94 and 2.05 GeV for the zero- and unit-strangeness states, respectively. These values are in good agreement with linear Gell-Mann–Okubo mass relations.

(c) The predicted $1D$ and $2S$ states in the $\bar{\mathfrak{N}}\mathfrak{N}$ system bracket the region of $\rho'(1600)$, $g(1680)$, and $A_3(1640)$. Allowing for some D, S mixing and including spin-orbit effects, we feel this agreement to be reasonable.

(d) For the $\bar{c}c$ system we predict a set of narrow-width $1P$ charmonium states around 3.5 GeV. A $2S$ $\bar{c}c$ state is predicted at 4.3 GeV. Mixing between $1D$ and $2S$ may change values for both states; in fact, such mixing must occur in order for the decay $\psi' \to e^+e^-$ to proceed with reasonable rates.[7] The $2S$ state may well be lowered to a region near the charm threshold. More details of the level splitting and transitions have been discussed by a number of authors.[8,9]

While this manuscript was being prepared, we received two very interesting papers on the spectrum of charmonium by Eichten *et al.*[9] and by Harrington, Park, and Yildiz.[10] In both cases a linear confinement potential is used.

TABLE I. Spectrum of low-lying meson states predicted by the model. Masses are given in units of GeV.

$(\bar{q}q)$	$M(1S)$	$M(1P)$	$M(1D)$	$M(2S)$
$(\bar{c}c)$	3.07	3.45	3.67	4.30
$(\bar{c}s)$	2.05	2.41	2.66	3.11
$(\bar{c}n)$	1.94	2.35	2.57	2.98
$(\bar{s}s)$	1.04	1.40	1.71	2.00
$(\bar{s}\mathfrak{N})$	0.92	1.31	1.70	2.07
$(\bar{\mathfrak{N}}\mathfrak{N})$	0.79	1.18	1.50	1.80

Note added.—After this paper had been submitted, we received a paper by Testa[11] in which the quark masses are calculated in terms of deep inelastic experimental data with the result $m_s^2 - m_u^2 \approx 0.08 \text{ GeV}^2$ in good agreement with ours.

[1]M. Gell-Mann, R. Oakes, and B. Renner, Phys. Rev. 175, 2195 (1968); S. L. Glashow and S. Weinberg, Phys. Rev. Lett. 20, 224 (1968).

[2]D. J. Gross and F. Wilczek, Phys. Rev. Lett. 30, 1343 (1971); H. D. Politzer, Phys. Rev. Lett. 30, 1346 (1973).

[3]T. Appelquist and H. D. Politzer, Phys. Rev. Lett. 34, 43 (1975).

[4]A. De Rújula and S. L. Glashow, Phys. Rev. Lett. 34, 47 (1975).

[5]It has been pointed out to us that the confinement potential proposed here resembles (in a very approximate sense) that of the bag model of A. Chodos *et al.*, Phys. Rev. D 9, 3471 (1974).

[6]See, for example, M. K. Gaillard, B. W. Lee, and J. L. Rosner, to be published. We shall on the main follow the particle nomenclature of these authors.

[7]A calculation of mixing based on positronium does not give agreement with the ψ' decay data. It is probably overoptimistic to expect such a calculation to represent tensor effects which would occur in a relativistic version of the model with confinement. However, this may indicate that a more plausible assignment for ψ' is $2S$.

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[9]E. Eichten, K. Gottfried, T. Kinoshita, J. Kogut, K. D. Lane, and T. M. Yan, Phys. Rev. Lett. 34, 369 (1975).

[10]B. J. Harrington, S. Y. Park, and A. Yildiz, Phys. Rev. Lett. 34, 168 (1975).

[11]M. Testa, California Institute of Technology Report No. 68-482, 1974 (to be published). The final results are obtained with the chiral condition $m_s \approx 25m_u$ superimposed.