Pattern of quark mass matrices and the mixing-angle hierarchy

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By expressing the quark masses, as well as the quark-mixing angles, in powers of the Cabibbo angle (λ) we show that the up- and down-quark mass matrices are proportional to each other up to terms of order λ^2. This implies that the mass ratio of these two types of quarks should be roughly the same in each generation except for the lightest quarks. We then demonstrate that this is the key ingredient that enables the Fritzsch ansatz for quark mass matrices to reproduce the observed pattern of Kobayashi-Maskawa (KM) mixings and predict that the KM angle $\theta_3=O(\lambda^3)$. Some of the phenomelogical implications, including that for the kaon CP-violation parameter ϵ, are worked out.

I. INTRODUCTION

Phenomenologically the $SU(3)\times SU(2)\times U(1)$ gauge theory of particle interactions with three generations of leptons and quarks has been very successful. On the more basic level, however, a certain theoretical question is still unanswered: Why are there three generations of fermions with identical gauge couplings? Will this sequential replication continue to yet another generation? Related to this deficiency, all fermion masses and mixing angles are free parameters of the theory. Clearly it will be desirable to have some simple mechanism that will account for the observed systematics of these parameters. Such study may turn out to be helpful for various formulations of generation symmetries, and it will aid in our search for the fourth generation if such leptons and quarks indeed exist.

In this paper our discussion will be confined to the three-generation case. A simple model with "maximal CP violation" allows us to express all the quark-mixing angles and phase in terms of the masses. Namely, the famous four-quark result[1] for the relation of the Cabibbo angle and quark masses,

$$\theta_C \simeq (m_d/m_s)^{1/2} , \tag{1}$$

is extended to all the Kobayashi-Maskawa[2] (KM) angles. In the accompanying paper[3] this simple model will be extended to the four-generation case. Some of the principal conclusions of these two papers have already been presented in a previous publication.[4]

This paper is organized as follows. In the next section, following Wolfenstein,[5] the quark mass and mixing-angle hierarchies are presented as series in powers of the Cabibbo angle (λ). It is then shown[6] in Sec. III that the up- and down-quark mass matrices are approximately proportional to each other with the difference of the (normalized) mass matrices being of the order λ^2. Thus we suggest that, apart from the light quarks, the ratio of these two masses in each generation should be roughly the same,[7]

$$m_{(2/3)}/m_{(-1/3)} = \text{generation independent} , \tag{2}$$

and in this way the case for $m_t \simeq 40$ GeV (Ref. 8) is favored. In Sec. IV we show that the Fritzsch ansatz[9] for quark mass matrices when combined with the above result and a particular version of maximal CP-violation postulate[10,11] will yield a set of KM quark-mixing angles $\delta=O(1)$, $\theta_1=O(\lambda)$, $\theta_2=O(\lambda^2)$ which are in agreement with the known phenomenology, and the prediction $\theta_3=O(\lambda^3)$ which leads to

$$\left|\frac{U_{ub}}{U_{cb}}\right| \simeq 0.06, \text{ and } \frac{\Gamma(b\to ue\bar{\nu})}{\Gamma(b\to ce\bar{\nu})} \simeq 0.006 . \tag{3}$$

Finally in Sec. V the above result is applied to the "box diagram" calculation of the kaon CP impurity parameter ϵ and it yields

$$\epsilon \simeq 1.9\times 10^{-3} B_K ,$$

where B_K is the conventional parameter of the $\Delta S=2$ short-distance operator between the K^0 and $\bar{K}^0$ states. (These numerical results are of course sensitive to all of the uncertainties in our knowledge of the quark masses.)

II. SYSTEMATICS OF FERMION MASSES AND MIXING ANGLES

The masses and mixing angles of the known quark states reveal some distinctive regularities. The following ones are most prominant.

Mass hierarchy. Just like the case of the charged leptons $m_e \ll m_\mu \ll m_\tau$ we also find that the mass ratios of the same charge particles in different generations all differ significantly from 1. The running masses at the energy scale of 1 GeV have been estimated to have the values[12]

$$\begin{aligned} &m_d \simeq 8.9\pm 2.6 \text{ MeV}, \quad m_s \simeq 175\pm 55 \text{ MeV} ,\\ &m_b \simeq 5.3\pm 0.1 \text{ GeV}, \\ &m_u \simeq 5.1\pm 1.5 \text{ MeV}, \quad m_c \simeq 1.35\pm 0.05 \text{ GeV} . \end{aligned} \tag{4}$$

And if we tentatively accept the interpretation of the $\bar{p}p$ collision data as indeed indicating the production of the t

quark,[8] we then also have

$$m_t = 30\text{--}50\ \text{GeV}\ . \tag{5}$$

Mixing-angle hierarchy. Representing the fact that the charged weak transitions take place dominately between quark states that are closest in their masses, the KM mixing matrix does not differ significantly from the unit matrix. Namely, intergeneration mixings are all small. Incorporating the recent experimental result on the B lifetime and $(b \to u)/(b \to c)$ branching-ratio limit, the magnitude of the KM elements are constrained to be[13]

$$U = \begin{bmatrix} 0.9733 \pm 0.0024 & 0.225 \pm 0.005 & < 0.009 \\ 0.24 \pm 0.03 & 0.82 \pm 0.13 & 0.058 \pm 0.009 \\ \cdots & \cdots & \cdots \end{bmatrix}, \tag{6}$$

where the rows are u, c, and t and the columns are d, s, and b. We note that the small off-diagonal elements also display a distinctive hierarchy.

To organize our thinking of the nine parameters in (6) with their disparate magnitudes, we shall parametrize them as powers of some common small parameter λ. Following Wolfenstein,[5] we choose to use the Cabibbo angle for λ,

$$U_{us} \simeq 0.225 \equiv \lambda\ , \tag{7}$$

and after imposing the requirement of unitarity, the KM matrix is parametrized, up to an $O(\lambda^4)$ correction, as

$$U = \begin{bmatrix} 1-\lambda^2/2 & \lambda & (\sigma - i\eta)A\lambda^3 \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ (1-\sigma-i\eta)A\lambda^3 & -A\lambda^2 & 1 \end{bmatrix}, \tag{8}$$

where $A \simeq 1.15$ and $(\sigma^2+\eta^2)^{1/2} < 0.7$.

In the same spirit we also choose to express the quark masses in (4) and (5) as powers of the same small parameter:

$$\begin{aligned} m_t : m_c : m_u &= 1 : c_t\lambda^2 : u_t\lambda^6\ , \\ m_b : m_s : m_d &= 1 : s_b\lambda^2 : d_b\lambda^4\ , \end{aligned} \tag{9}$$

where we have the $O(1)$ coefficients

$$c_t = (m_c/m_t)/\lambda^2,\quad u_t = (m_u/m_t)/\lambda^6\ ,$$

$$s_b = (m_s/m_b)/\lambda^2,\quad d_b = (m_d/m_b)/\lambda^4\ .$$

We wish to emphasize that this is merely a parametrization of the magnitudes of quark masses and it has no particular physical significance. The regularity unveiled by this parametrization is that the mass of each succeeding generation increases by $\lambda^{-2} \approx 20$. This pattern is broken only by the "anomalously light" u quark.

III. PATTERN OF QUARK MASS MATRIX ELEMENTS

In gauge theory the more fundamental theoretical object is the mass matrices defined with respect to fermion states that carry definite gauge interaction quantum numbers. When diagonalized to get the physical mass eigenstates the matrix eigenvalues yield the mass spectra and the product of the diagonalizing unitary transformations yield the mixing angles and phases in the charged weak couplings. Thus it is these mass matrices that we shall direct our attention to.

The purpose of this paper is to show that the systematics of the observed quark masses and mixing angles is consistent with the hypothesis that the up- and down-quark mass matrices are closely proportional to each other. In this section we shall first use the observed mass and mixing-angle hierarchies to show that the difference between the (normalized) mass matrices $(M_d/m_b - M_u/m_t)$ is indeed small, of the order λ^2.

In this section we shall take the mass matrices to be Hermitian. In the standard $SU(3)\times SU(2)\times U(1)$ model this can always be done through a redefinition of the right-handed quark fields which are gauge singlets. Beyond the standard model, there are many interesting models in which the Hermiticity of mass matrices can be naturally obtained. Notably this is the case for the left-right-symmetric models and, thus, by extension, for the important grand-unified-theory (GUT) models of O(10) and E(6). For the more general situation one can always carry out the following analysis for the Hermitian product of $M^\dagger M$ instead of mass matrix M itself.[14]

The Hermitian mass matrices in the generation spaces, M_u and M_d, can be diagonalized by unitary transformations (instead of biunitary transformations if the Hermiticity condition is removed):

$$\begin{aligned} V^u M_u V^{u\dagger} &= \hat{M}_u\ , \\ V^d M_d V^{d\dagger} &= \hat{M}_d\ , \end{aligned} \tag{10}$$

where $\hat{M}_u$ and $\hat{M}_d$ are the diagonalized up- and down-quark mass matrices, respectively. The KM matrix U is simply the product of the unitary transformations

$$U = V^u V^{d\dagger}\ . \tag{11}$$

Phenomenologically the KM matrix [see Eqs. (6) and (8)] is close to the unit matrix; the only nondiagonal elements that deviate significantly from zero being U_{us} and U_{dc} (the Cabibbo angle). Through Eq. (11) this implies that the two unitary matrices V^u and V^d are approximately equal. Namely, M_u and M_d almost commute with each other so that they can be diagonalized by the same unitary matrix. In fact we shall make a stronger postulate: this commutivity is achieved simply by having M_u and M_d being closely proportional to each other,

$$M_d = xM_u + \Delta'\ , \tag{12}$$

where x is a constant and the matrix Δ' is a small correction. In the limit of strict proportionality $(\Delta'=0)$, we would have $V^d = V^u$ and, thus, $U=1$. A direct implication of this postulate is that the ratio of the up- to the down-quark masses is the same for each generation:

$$m_u/m_d = m_c/m_s\ , \tag{13a}$$

$$m_t/m_b = m_c/m_s\ . \tag{13b}$$

m_c/m_s being close to 8 indicates that Eq. (13b) is valid for the central value of m_t (i.e., 40 GeV). We shall assume that it is not an accident and take it as supporting this interpretation of $U \simeq 1$. The fact that Eq. (13a) is badly violated can be accounted for by the smallness of first-generation mass values. Namely, it is expected that m_u and m_d receive significant contribution from the correction term Δ. If we define the "normalized mass matrices"

$$\overline{M}_d = M_d/m_b, \quad \overline{M}_u = M_u/m_t ,$$

a priori their difference can be of the order λ:

$$\Delta = \overline{M}_d - \overline{M}_u = O(\lambda) .$$

However, with the mass and mixing-angle hierarchies as given in Eqs. (8) and (9), we can show that this difference matrix is even smaller: $\Delta = O(\lambda^2)$. To demonstrate this, we shall work in the basis where M_u is diagonal:

$$\Delta = U\hat{\overline{M}}_d U^\dagger - \hat{\overline{M}}_u . \tag{14}$$

[This equation is obtained by making unitary transformation on both sides of (12) and set $x = m_b/m_t$.] The information as contained in (8) and (9) can be inserted into Eq. (14) with (for simplicity of presentation we shall drop the imaginary part)

$$\begin{aligned} &U = 1 + \lambda T_1 + \lambda^2 T_2 + \lambda^3 T_3 + \cdots , \\ &\hat{\overline{M}}_d = \hat{b}_b - \lambda^2 \hat{s}_b + \lambda^4 \hat{d}_b , \\ &\hat{\overline{M}}_u = \hat{t}_t - \lambda^2 \hat{c}_t + \lambda^6 \hat{u}_t , \end{aligned} \tag{15}$$

where

$$\begin{aligned} &T_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & A \\ 0 & -A & 0 \end{bmatrix}, \\ &T_3 = \begin{bmatrix} 0 & 0 & A\sigma \\ 0 & 0 & 0 \\ A(1-\sigma) & 0 & 0 \end{bmatrix}, \end{aligned} \tag{16}$$

and

$$\begin{aligned} &\hat{b}_b = \mathrm{diag}(0,0,1), \quad \hat{s}_b = \mathrm{diag}(0,s_b,0) , \\ &\hat{d}_b = \mathrm{diag}(d_b,0,0), \quad \hat{t}_t = \mathrm{diag}(0,0,1) , \\ &\hat{c}_t = \mathrm{diag}(0,c_t,0), \quad \hat{u}_t = \mathrm{diag}(u_t,0,0) , \end{aligned} \tag{17}$$

where s_b, c_t, d_b, and u_t are defined in Eq. (9).

We then have

$$\begin{aligned} \Delta = &(1 + \lambda T_1 + \lambda^2 T_2 + \lambda^3 T_3)(\hat{b}_b - \lambda^2 \hat{s}_b + \lambda^4 \hat{d}_b) \\ &\times (1 + \lambda T_1^\dagger + \lambda^2 T_2^\dagger + \lambda^3 T_3^\dagger) - (\hat{t}_t - \lambda^2 \hat{c}_t + \lambda^6 \hat{u}_t) . \end{aligned}$$

The zeroth order is trivially satisfied [i.e., the right-hand side (RHS) vanishes] because we have already put in the consistency condition of (12) for the proportional constant $x = m_b/m_t$. The linear λ term also vanishes,

$$\lambda(T_1 \hat{b}_b + \hat{b}_b T_1^\dagger) = 0 , \tag{18}$$

because the T_1 matrix has nonzero elements only in the first two columns and rows while $\hat{b}_b$ vanishes for these entries. This is related to the input that U_{cb} and U_{ts} are of the order λ^2 (rather than linear in λ as one would at first expect). Thus, Δ is actually at least quadratic in λ:

$$\Delta = \begin{bmatrix} 0 & -s_b\lambda^3 & A\sigma\lambda^3 \\ -s_b\lambda^3 & (c_t - s_b)\lambda^2 & A\lambda^2 \\ A\sigma\lambda^3 & A\lambda^2 & 0 \end{bmatrix} . \tag{19}$$

This result follows directly from the parametrization of masses and KM angles in powers of λ. No further relation among the coefficients of expansion has been assumed, and thus, s_b, c_t, A, and σ are all supposed to be independent coefficients of order 1. However, we may wish to keep in mind that the zeroth-order relation (13b) and the definitions in Eq. (9) imply that $c_t - s_b = O(\lambda)$. Equation (19) informs us that the difference of the normalized mass matrices has elements with order of magnitude as

$$\Delta = \begin{bmatrix} <\lambda^3 & \lambda^3 & \lesssim\lambda^3 \\ \lambda^3 & \lesssim\lambda^2 & \lambda^2 \\ \lesssim\lambda^3 & \lambda^2 & 0 \end{bmatrix} . \tag{20}$$

What are the implications for the mass matrices from Eq. (20)? The largest element of the normalized matrices $\overline{M}_u$ and $\overline{M}_d$ is, by construction, 1, the smaller elements are of the order $O(\lambda)$, as one would expect from the input of $U = 1 + O(\lambda)$. However, now we know that $\Delta = O(\lambda^2)$, we have two plausible possibilities for the mass matrices.

Case (i). The small elements of $\overline{M}_u$ and $\overline{M}_d$ are at most of the same order as those in Δ:

$$\overline{M}_{u,d} = \begin{bmatrix} <\lambda^3 & \lambda^3 & \lesssim\lambda^3 \\ \lambda^3 & \lesssim\lambda^2 & \lambda^2 \\ \lesssim\lambda^3 & \lambda^2 & 1 \end{bmatrix} \tag{21}$$

and no precise cancellation is needed to obtain a difference matrix Δ of the order as given in (20).

Case (ii). The small elements of the $\overline{M}_u$ and $\overline{M}_d$ can be of the order λ:

$$\overline{M}_{u,d} = \begin{bmatrix} <\lambda^3 & \lambda^3 & \lesssim\lambda^3 \\ \lambda^3 & \lesssim\lambda^2 & \lambda \\ \lesssim\lambda^3 & \lambda & 1 \end{bmatrix} \tag{22}$$

and the λ terms cancel resulting in a λ^2 term in the difference.

In a certain sense case (i) is more natural; no fine-tuning is required whatsoever. However, it turns out that case (ii) is more interesting. Its form naturally suggests the Fritzsch-type matrices, and as we shall see the required cancellation comes out rather naturally.

IV. FRITZSCH-TYPE MASS MATRIX

A notable example of the case (ii) mass matrices is the one written by Fritzsch.[9] This ansatz states that only the heaviest generation has a diagonal element and all other lighter masses arise through mixings between neighboring families: We have, for $a = u,d$,

$$M_a=\begin{bmatrix} 0 & A_a e^{i\alpha_a} & 0 \\ A_a e^{i\alpha'_a} & 0 & B_a e^{i\beta_a} \\ 0 & B_a e^{i\beta'_a} & C_a e^{i\gamma_a} \end{bmatrix},$$

which can be written as $M_a=P_aF_aQ_a$, with

$$F_a=\begin{bmatrix} 0 & A_a & 0 \\ A_a & 0 & B_a \\ 0 & B_a & C_a \end{bmatrix} \tag{23}$$

being a real symmetric matrix and P_a and Q_a are diagonal phase matrices. M_a has the structure that allows one to express the resultant mixing angles in terms of the eigenvalues and the phases of the mass matrix. F_a can be diagonalized by orthogonal transformations O_a:

$$O_aF_aO_a^T=\hat{M}_a \ . \tag{24}$$

The KM matrix, being the product of the two (left-handed) unitary transformations $V_L^u V_L^{d\dagger}=O^u P_u P_d^* O^{dT}$, can be written as

$$U=XO^uPO^{dT}Y \ , \tag{25}$$

where the diagonal phase matrix P is the product $P_uP_d^*$, and the other two matrices X and Y are inserted to represent the rephasing freedom of the KM elements through redefinition of the quark phases. Since, for each $a=u$ and d, the Fritzsch matrix F has only three nonzero elements A, B, and C, they can be expressed in terms of the three eigenvalues $\hat{M}=\text{diag}(m_1,-m_2,m_3)$ by equating the invariants (e.g., the trace and determinant):

$$A=\left[\frac{m_3m_2m_1}{m_3-m_2+m_1}\right]^{1/2}\simeq(m_2m_1)^{1/2} \ ,$$

$$B=\left[\frac{(m_3-m_2)(m_3+m_1)(m_2-m_1)}{m_3-m_2+m_1}\right]^{1/2}\simeq(m_3m_2)^{1/2} \ , \tag{26}$$

$$C=(m_3-m_2+m_1)\simeq m_3 \ ,$$

where the approximate equalities on the RHS are valid for the hierarchical order of $m_1\ll m_2\ll m_3$. Namely, the magnitude of the up- and down-quark mass matrix elements are given by the Fritzsch ansatz to be

$$\overline{M}_u=\begin{bmatrix} 0 & (m_u/m_c)^{1/2}(m_c/m_t) & 0 \\ (m_u/m_c)^{1/2}(m_c/m_t) & 0 & (m_c/m_t)^{1/2} \\ 0 & (m_c/m_t)^{1/2} & 1 \end{bmatrix},$$
$$\overline{M}_d=\begin{bmatrix} 0 & (m_d/m_s)^{1/2}(m_s/m_b) & 0 \\ (m_d/m_s)^{1/2}(m_s/m_b) & 0 & (m_s/m_b)^{1/2} \\ 0 & (m_s/m_b)^{1/2} & 1 \end{bmatrix}. \tag{27}$$

However, in order that their difference coming out has the magnitude as given in Eq. (20), there must be cancellations for the $\overline{M}_{23}$ and $\overline{M}_{32}$ elements while not for the $\overline{M}_{12}$ and $\overline{M}_{21}$. But this is precisely the situation one would expect for an approximate proportionality of M_u and M_d as discussed above. The limiting relation (2) is broken differently for the light and the heavy quarks. That Eq. (13a) is poorly obeyed can be expressed as

$$(m_d/m_s)^{1/2}-(m_u/m_c)^{1/2}=O(\lambda) \ . \tag{28a}$$

That Eq. (13b) is approximately valid means that

$$(m_s/m_b)^{1/2}-(m_c/m_t)^{1/2}=O(\lambda^2) \ . \tag{28b}$$

(Namely, here the difference is much smaller than the individual ratio.) Thus Eq. (20) for the difference matrix Δ is obtained because the up- and down-quark mass matrices differ significantly from each other only for those elements involving the first generation. In terms of magnitude, the relation of $\overline{M}_d-\overline{M}_u=\Delta$ can be written as

$$\begin{bmatrix} 0 & \lambda^3 & 0 \\ \lambda^3 & 0 & \lambda \\ 0 & \lambda & 1 \end{bmatrix}-\begin{bmatrix} 0 & \lambda^4 & 0 \\ \lambda^4 & 0 & \lambda \\ 0 & \lambda & 1 \end{bmatrix}=\begin{bmatrix} 0 & \lambda^3 & 0 \\ \lambda^3 & 0 & \lambda^2 \\ 0 & \lambda^2 & 0 \end{bmatrix}.$$

In the following we shall demonstrate more explicitly that the mass relations (28a) and (28b) implied by the approximate proportionality of M_u and M_d are just the key ingredients needed for the Fritzsch ansatz to reproduce the observed pattern of mixing hierarchy (8).

Since the Fritzsch matrix elements can be expressed in terms of the masses as in Eq. (26), the diagonalization procedure will lead to a KM matrix with elements expressible in terms of the mass eigenvalues and phases. It has already been shown[10] that compatibility with the observed mixing angles can be achieved by an adjustment of m_t and the phases. The range of possible values includes $m_t\simeq 40$ GeV and a maximal CP-violation hypothesis.[10,11] Our approach will be to make systematic expansion of the matrix elements (in λ) and the resulting simplified expression involving only the leading terms will show clearly that the underlying pattern necessary for the success of this approach is just the approximate proportionality of M_u to M_d.

Start with the exact expression[15] for the orthogonal matrix that diagonalizes the Fritzsch mass matrix with elements given by (26):

$$O_{11}=\left[\frac{m_3 m_2(m_3-m_2)}{(m_3-m_1)(m_2+m_1)(m_3-m_2+m_1)}\right]^{1/2},$$

$$O_{21}=-\left[\frac{m_3 m_1(m_3+m_1)}{(m_3+m_2)(m_2+m_1)(m_3-m_2+m_1)}\right]^{1/2},$$

$$O_{31}=\left[\frac{m_2 m_1(m_2-m_1)}{(m_3+m_2)(m_3-m_1)(m_3-m_2+m_1)}\right]^{1/2},$$

$$O_{12}=\left[\frac{m_1(m_3-m_2)}{(m_3-m_1)(m_2+m_1)}\right]^{1/2},$$

$$O_{22}=\left[\frac{m_2(m_3+m_1)}{(m_3+m_2)(m_2+m_1)}\right]^{1/2}, \quad (29)$$

$$O_{32}=\left[\frac{m_3(m_2-m_1)}{(m_3+m_2)(m_3-m_1)}\right]^{1/2},$$

$$O_{13}=-\left[\frac{m_1(m_3+m_1)(m_2-m_1)}{(m_3-m_1)(m_2+m_1)(m_3-m_2+m_1)}\right]^{1/2},$$

$$O_{23}=-\left[\frac{m_2(m_3-m_2)(m_2-m_1)}{(m_3+m_2)(m_2+m_1)(m_3-m_2+m_1)}\right]^{1/2},$$

$$O_{33}=\left[\frac{m_3(m_3-m_2)(m_3+m_1)}{(m_3-m_2+m_1)(m_3+m_2)(m_3-m_1)}\right]^{1/2}.$$

Thus for the hierarachical masses $m_3 \gg m_2 \gg m_1$ we have

$$O=\begin{bmatrix} 1-(m_1/2m_2) & (m_1/m_2)^{1/2} & -(m_1/m_3)^{1/2} \\ -(m_1/m_2)^{1/2} & 1-(m_1/2m_2)-(m_2/2m_3) & -(m_2/m_3)^{1/2} \\ (m_1/m_3)^{1/2}(m_2/m_3) & (m_2/m_3)^{1/2} & 1-(m_2/2m_3) \end{bmatrix}. \quad (30)$$

This is to be applied for both O^u and O^d, and the KM mixings can then be obtained from Eq. (25). Immediately we note that for the simple case where all phases are set to zero, Eqs. (25) and (30) yield

$$U_{us}=(m_u/m_c)^{1/2}-(m_d/m_s)^{1/2}, \quad (31a)$$

$$U_{cb}=-(m_c/m_t)^{1/2}+(m_s/m_b)^{1/2}. \quad (31b)$$

Through Eqs. (28a) and (28b) we see that they have just the right magnitudes as in (8). We will now demonstrate that a simple choice of phases can be made to preserve this desired cancellation together with the attractive possibility of maximal *CP* violation.

Start with the definition of phase matrix elements:

$$X=\operatorname{diag}(e^{i(\alpha_k-\beta_k)}),$$

$$Y=\operatorname{diag}(e^{-i\alpha_k}),$$

$$P=\operatorname{diag}(e^{i\gamma_k});$$

Eq. (25) has the components

$$U_{ij}=\left[\sum_k O^u_{ik}O^d_{jk}e^{i\gamma_k}\right]e^{-i\beta_i}e^{i(\alpha_i-\alpha_j)}. \quad (32)$$

(A) Convention. The X and Y matrices represent the rephasing freedom of the KM elements through the redefinition of quark phases. Thus the choice of α_k and β_k is a matter of convention. We pick $\alpha_1=\alpha_2+\pi$ and $\alpha_2=\alpha_3$ so that the overall phase factor $e^{i(\alpha_i-\alpha_j)}$ at each (i,j) position is either $+1$ or -1. For the diagonal terms

$$U_{ud}=(e^{i\gamma_1}+\cdots)e^{-i\beta_1},$$

$$U_{cs}=(e^{i\gamma_2}+\cdots)e^{-i\beta_2},$$

$$U_{tb}=(e^{i\gamma_3}+\cdots)e^{-i\beta_3},$$

where $\cdots$ represent subleading $O(\lambda^2)$ terms, our choice of $\beta_k=\gamma_k$ will result in having the dominant term being kept real.

(B) Assumption. The γ_k phases originate from M_u and M_d. Therefore what we choose for their value will reflect our assumption about the phase structure of the quark mass matrices. The off-diagonal elements of (31) now read

$$U_{us}=(m_d/m_s)^{1/2}-(m_u/m_c)^{1/2}e^{i(\gamma_2-\gamma_1)}, \quad (33a)$$

$$U_{cb}=(m_s/m_b)^{1/2}-(m_c/m_t)^{1/2}e^{i(\gamma_3-\gamma_2)}. \quad (33b)$$

To maintain the good cancellation of Eq. (31b) in (33b) requires the choice of $\gamma_2=\gamma_3$. On the other hand, there is no cancellation in (33a) to require $\gamma_1=\gamma_2$. In fact $\gamma_1=\gamma_2$ would imply a totally real KM matrix and no *CP* violation, it naturally suggests that we choose the maximal phase[10,11] of

$$\gamma_1=\gamma_2-\pi/2. \quad (34)$$

(That $\gamma_1=\gamma_2+\pi/2$ is equally plausible and we shall return to the question of the sign at a later stage.) Such a phase could, for example, be obtained with a real M_d and an up-quark mass matrix $\overline{M}_u$ which is proportional to $\overline{M}_d$ up to $O(\lambda^2)$ with the phase associated to the small λ^4 term:

$$\overline{M}_d=\begin{bmatrix} 0 & \lambda^3 & 0 \\ \lambda^3 & 0 & \lambda \\ 0 & \lambda & 1 \end{bmatrix}, \quad (35)$$

$$\overline{M}_u=\begin{bmatrix} 0 & i\lambda^4 & 0 \\ -i\lambda^4 & 0 & \lambda \\ 0 & \lambda & 1 \end{bmatrix}. \quad (36)$$

Namely, $P_d = Q_d = 1$, $P_u = \text{diag}(-i,1,1)$, and $Q_u = \text{diag}(i,1,1)$. We should point out that such a phase choice is also favored by phenomenology: all indications in kaon *CP*-violation phenomenology point towards the need for a large *CP*-violation phase. (See Sec. V.)

We are now ready to compare our result with the observed pattern as summarized in the Wolfenstein parametrization (8). In principle one should first make a rephasing operation to make U_{us} real as in (8). However, since the phase ϕ in

$$U_{us} = (m_d/m_s)^{1/2} - i(m_u/m_c)^{1/2} \tag{37}$$

is small

$$\phi = \arctan(m_u m_s/m_c m_d)^{1/2} = O(\lambda) , \tag{38}$$

we can simply ignore this rephasing effect in our calculation of the KM parameters to the leading power in λ. In this way, from

$$U_{cb} = (m_s/m_b)^{1/2} - (m_c/m_t)^{1/2} \equiv A\lambda^2 , \tag{39}$$

$$U_{ub} = -(m_d/m_b)^{1/2}(m_s/m_b) - iA\lambda^2(m_u/m_c)^{1/2}$$
$$\equiv A\lambda^3(\sigma - i\eta) \tag{40}$$

we find

$$\sigma \simeq -(m_s/m_b)^{3/2}/|U_{cb}| \simeq -0.10 , \tag{40a}$$

$$\eta \simeq (m_u m_s/m_c m_d)^{1/2} \simeq 0.27 . \tag{40b}$$

With respect to the results (39) and (40), we make the following comments.

(i) In principle the Wolfenstein parameters σ and η can be of the order 1. However, Eq. (40) shows that they are actually of the order λ.

(ii) In terms of the more familiar KM angles, our results read ($s_i \equiv \sin\theta_i$)

$$s_1 = \lambda \simeq 0.225 ,$$
$$s_2 = A\lambda^2[(1-\sigma)^2 + \eta^2]^{1/2} \simeq A\lambda^2 \simeq |U_{cb}| \simeq 0.058 , \tag{41}$$
$$s_3 = A\lambda^2(\sigma^2 + \eta^2)^{1/2} \simeq 0.017 ,$$

and, from $s_2 s_3 \sin\delta = A^2\lambda^4\eta$,

$$\sin\delta \simeq s_2\eta/s_3 \simeq [1 + (\sigma/\eta)^2]^{-1/2} \simeq 0.93 . \tag{42}$$

Thus KM angles all correspond to different orders of λ: $\sin\delta = O(1)$, $s_1 = O(\lambda)$, $s_2 = O(\lambda^2)$, and $s_3 = O(\lambda^3)$. In other words the famous two-generation result of expressing the Cabibbo angle in terms of quark mass ratio $\theta_C \simeq (m_d/m_s)^{1/2}$ can be generalized to the situation for three generations:

$$\begin{aligned} &\theta_1 \simeq (m_d/m_s)^{1/2} , \\ &\theta_2 \simeq (m_s/m_b)^{1/2} - (m_c/m_t)^{1/2} , \\ &\theta_3 \simeq [(m_s/m_b)^3 + \theta_2^2(m_u m_s/m_c m_d)]^{1/2} , \\ &\sin\delta \simeq [1 + (m_c m_d m_s^2)/(m_u m_b \theta_2^2)]^{-1/2} . \end{aligned} \tag{43}$$

(iii) The sign of the parameter A clearly depends on the relative magnitude of (m_s/m_b) and (m_c/m_t). This is also the case for σ as its definition involves A. On the other hand, the sign of η is sensitive to whether the mass matrix phase $\gamma_2 - \gamma_1 = \gamma_3 - \gamma_1$ is plus or minus $\pi/2$. Had we picked it to be $-\pi/2$ instead of Eq. (34), the resultant sign for η would be reversed also. And this will affect the calculation of *CP*-violation parameters.

(iv) The immediate "experimental" implication of our result

$$|U_{ub}/U_{cb}| \simeq \lambda(\sigma^2 + \eta^2)^{1/2} \simeq 0.06$$

is a prediction for the quark branching ratio:

$$\frac{\Gamma(b \to ue\bar{\nu})}{\Gamma(b \to ce\bar{\nu})} = (1 - 8x^2 + 8x^6 - x^8 - 24x^4 \ln x)^{-1} \left| \frac{U_{ub}}{U_{cb}} \right|^2 \simeq 0.006$$

if the quark mass values quoted in Eq. (5a) are used for $x = m_c/m_b = 0.255$. This value for the branching ratio is to be compared to the present experimental upper limit of 0.04.

(v) One should keep in mind that all of the above are results in the leading λ approximation. They are uncertain up to some $\lambda = 20\%$ corrections, as well as the uncertainties in the quark mass values used in their calculations.

V. *CP* VIOLATION IN THE K^0-$\bar{K}^0$ SYSTEM

Finally as an application of the KM mixing-angle result of Sec. IV we calculate the kaon *CP* impurity parameter ϵ in the six-quark standard model. ϵ is dominated by the short-distance physics as represented by the box diagrams involving the exchange of heavy quarks:[16]

$$|\epsilon| \simeq \left[\frac{G_F^2 f_k^2 m_c^2}{\sqrt{2}12\pi^2} \frac{m_k}{\Delta m_k} \right] B_K[\eta_{cc}\,\text{Im}\lambda_c^2 + \eta_{tt}(m_t^2/m_c^2)\,\text{Im}\lambda_t^2 + \eta_{ct}\ln(m_t^2/m_c^2) 2\,\text{Im}\lambda_c\lambda_t] , \tag{44}$$

where the expression in the large parentheses yields a value of 5.52 corresponding to the experimental numbers of $G_F = 1.18 \times 10^{-5}$ GeV^{-2}, $f_k = 0.16$ GeV, and $(\Delta m_k/m_k) = 0.7 \times 10^{-14}$. B_K is the well-known parameter of the $\Delta S = 2$ short-distance operator between the K^0 and $\bar{K}^0$ states. At present it is a major source of uncertainty in any such calculation. We have, for example, $B_K \simeq 0.33$ from current algebra[17] and $B_K = 1.0$ from "vacuum saturation."[18] In Eq. (44) $\eta_{cc} \simeq 0.7$, $\eta_{tt} \simeq 0.6$, and $\eta_{ct} \simeq 0.4$ are the QCD correction factors.[19] The λ_i's are products of KM quark-mixing angles $\lambda_i = U_{is}^* U_{id}$ which, given the result of Sec. IV, we can now calculate. Since what is needed in Eq. (44) is the imaginary part of various products $U_{is}^* U_{id} U_{js}^* U_{jd}$, we must extend the parametrization of (8) to include the leading terms in the imaginary as well as the real parts. As we have the freedom to choose our phase convention so that five of the KM elements are real (which we pick them to be U_{ud}, U_{us}, U_{cd}, U_{ts}, and U_{tb}), only two subleading imaginary parts (those of U_{cs} and U_{cb}) need to be determined. From the unitarity condition, such as

$$\sum_i U_{ci}^* U_{ti} = \sum_i U_{ci}^* U_{ui} = 0 ,$$

we find

$$\mathrm{Im}\, U_{cs} = -A^2\eta\lambda^4 \text{ and } \mathrm{Im}\, U_{cb} = A\eta\lambda^4 .$$

The leading real and imaginary parts of λ_c and λ_t are fixed to be $-\lambda - iA^2\eta\lambda^5$ and $-A^2\lambda^5 + iA^2\eta\lambda^5$, respectively. Thus all of the mixing-angle factors in the box diagram are determined:

$$\mathrm{Im}\lambda_c{}^2 = -2\,\mathrm{Im}\lambda_c\lambda_t = 2A^2\eta\lambda^6 ,$$

$$\mathrm{Im}\lambda_t{}^2 = -2A^4\eta\lambda^{10} .$$

This leads to an evaluation,

$$|\epsilon| \simeq 11.0A^2\eta\lambda^6[-0.7 + 0.6A^2\lambda^4(m_t{}^2/m_c{}^2) + 0.4\ln(m_t{}^2/m_c{}^2)]B_K$$

$$\simeq 0.5\times 10^{-3}(-0.7 + 1.79 + 2.71)B_K$$

$$\simeq 1.9\times 10^{-3} B_K .$$

Namely, the standard six-quark model can account for the observed $|\epsilon| = 2.27\times 10^{-3}$ with a $B_K \simeq 1.2$.

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