The Kaluza–Klein theory and extra dimensions

17

- Einstein famously spent the latter half of his physics working life on his program of unified field theories. His conviction was that a unification program not only could combine his GR gravitational field theory with Maxwell's equations but also shed light on the quantum mystery. In this chapter, we discuss the Kaluza–Klein (KK) theory which, in many ways, is a shining example of Einstein's unification program. It has stayed relevant even for physics research in the twenty-first century. However, the Kaluza–Klein theory uses the more conventional quantum idea and does not illuminate its origin as Einstein had envisioned for a unified theory.
- The Kaluza–Klein theory is a GR field theory in a spacetime with an extra spatial dimension. The nonobservation of the extra fifth dimension is assumed to result from its compact size. (The extra dimension is curled up.) The theory not only achieves a unification of gravitation with electrodynamics but also suggests a possible interpretation of the charge space and gauge symmetry as reflecting the existence of this compactified extra dimension.
- By way of a long calculation, Theodor Kaluza has shown that the 5D general relativity field equation with a particular geometry is composed of two parts, one being the Einstein field equation and the other the Maxwell equation. This remarkable discovery has been called "the Kaluza–Klein miracle". The details of this calculation are provided in the SuppMat Section 17.5. We also discuss the motivation and meaning of the assumed geometry for this 5D spacetime.
- As gauge symmetry was being developed, Oskar Klein showed that a gauge transformation could be identified with a displacement in the extra dimension coordinate in the KK theory. This went a long way in explaining the KK "miracle". Also it was demonstrated that the relativistic Klein–Gordon wave equation in KK spacetime is equivalent to a set of decoupled 4D Klein–Gordon equations for a tower of particles with increasing masses. Thus the signature of an extra dimension is the existence of a tower of KK particles, having identical spin

and gauge quantum numbers, with increasing masses controlled by the compactification scale.

- In the last section brief comments are offered of the more recent efforts in the construction of unified theories with extra dimensions.

17.1 Unification of electrodynamics and gravity

17.1.1 Einstein and unified field theory

Unifying different realms of physics has always led to fresh insight into our physical world. Maxwell's and Faraday's fusion of electricity and magnetism brought new understanding of light and radiation. Einstein's motivation for new physics was often prompted by the promise of wider comprehension that a new synthesis would bring. Recall his motivation for the principle of relativity. His special relativity brought about the deep cognizance that space and time were interchangeable. The resultant insight that spacetime was the arena in which physical events took place ultimately brought about his geometric theory of gravitation—in the form of a dynamical spacetime. Concurrently extending his atomic hypothesis of matter (doctoral thesis and Brownian motion) as well to radiation, his light quantum idea, he found that the electromagnetic field could have the puzzling feature of being both wave and particle at the same time. As we have discussed in Chapter 8, while Einstein appreciated the specific successes of the new quantum mechanics, he could not believe it as an acceptable description of reality. It was in this context that Einstein had hopes of finding a unification of electrodynamics and general relativity that would also shed light on the quantum mystery. This was the driving force behind his 20-year effort in the unified field theory program.

17.1.2 A geometric unification

Einstein's accomplishment in formulating a geometric theory of gravitation naturally led him, and others, in efforts to find a geometric formulation of Maxwell's theory. As mentioned in Section 16.5.1, this was the original motivation of Hermann Weyl and it eventually led to fruition in the form of interpreting electrodynamics as a gauge interaction. If a more direct geometric formulation of electromagnetism is possible, it would perhaps make the unification with gravity more likely. As we shall see in this chapter, a geometric theory of electrodynamics was actually obtained through a unification attempt, but it is the geometry of a spacetime with an extra dimension.

While gauge symmetry bears a resemblance to general relativity of being also a local symmetry, the result still does not seem like much help in finding a unified theory. Gauge invariance being a local symmetry, not in ordinary spacetime, but in the internal charge space, the question naturally arises: what exactly is this charge space? A possible answer was found[1] in 1919 by the Prussian mathematician Theodor Kaluza (1885–1954). He suggested extending the general principle of relativity to a hypothetical 5D spacetime—the usual

[1] Kaluza's paper was sent to Einstein in 1919, but did not come out in print until two years later (Kaluza 1921).

4D spacetime augmented by an extra spatial dimension.[2] It was discovered that a particular restricted 5D space geometry would lead to a 5D general theory of relativity with a field equation composed of two parts, one being the Einstein equation and another being the Maxwell equation.

The possibility of such a ‘miraculous’ unification was explained by the Swedish physicist Oskar Klein (1894–1977) in the late 1920s when quantum mechanics and gauge theory were being developed. Klein showed that, in Kaluza’s 5D theory, a gauge transformation had the geometric significance of being a displacement in the extra dimension. Thus charge space has the physical meaning of being the extra space dimension. If this extra dimension is compactified (so as to have avoided direct detection), quantum theory predicts the existence of a tower of particles with ever increasing masses with the mass difference controlled by the compactification length size. This unified field theory has come to be called the Kaluza–Klein (KK) theory. However, the KK theory made use of the more conventional quantum idea and did not shed light on its origin as Einstein had envisioned for the unified theory.

The Kaluza–Klein theory postulates the existence of an extra spatial dimension. Our spacetime is actually five-dimensional; it was demonstrated that electromagnetism can be viewed as part of 5D general relativity. That is, one can “derive” electrodynamics by postulating the principle of general relativity in a 5D spacetime. To prepare for the study of this embedding of electromagnetic gauge theory in a 5D general relativity, we first recall the relevant parts of gauge symmetry as well as of the GR theory.

[2] Just about all attempts of extending spacetime dimensions involve extra spatial dimensions. The need to have the extra dimensions compactified would have to, in the case of an extra time dimension, overcome the serious causality difficulties associated with looped times.

17.1.3 A rapid review of electromagnetic gauge theory

Under a $U(1)$ gauge transformation, the 4-vector potential (regarded as the fundamental electromagnetic field) transforms as

$$A_\mu \longrightarrow A'_\mu = A_\mu + \partial_\mu \theta \tag{17.1}$$

where $\theta(x)$ is the gauge function, cf. Eq. (16.45). The EM field intensity, being the 4-curl of the gauge field

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{17.2}$$

is clearly invariant under the gauge transformation of (17.1). Requiring the Lagrangian density to be a relativistic and gauge invariant scalar leads to a free Maxwell density of

$$\mathcal{L}_{\text{EM}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{17.3}$$

As discussed in Sections 16.4.2 and 16.4.3, the Euler–Lagrange equation based on this $\mathcal{L}_{\text{EM}}$ is the Maxwell equation. In this sense we say Maxwell’s theory is essentially determined by (special) relativity and $U(1)$ gauge symmetry.

17.1.4 A rapid review of general relativistic gravitational theory

GR equations are covariant under the general coordinate transformation that leaves invariant the spacetime interval

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu,$$

where $g_{\mu\nu}$ is the metric tensor of the 4D spacetime. Just as the metric can be interpreted as a relativistic gravitational potential, the Christoffel symbols

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left[\partial_\nu g_{\mu\rho} + \partial_\mu g_{\nu\rho} - \partial_\rho g_{\mu\nu} \right], \tag{17.4}$$

being the first derivative of the potential [cf. Eq. (13.6)] can be thought as the gravitational field intensities. The Riemann–Christoffel curvature tensor [cf. Eq. (14.9)]

$$R^\mu_{\lambda\alpha\beta} = \partial_\alpha \Gamma^\mu_{\lambda\beta} - \partial_\beta \Gamma^\mu_{\lambda\alpha} + \Gamma^\mu_{\nu\alpha} \Gamma^\nu_{\lambda\beta} - \Gamma^\mu_{\nu\beta} \Gamma^\nu_{\lambda\alpha}, \tag{17.5}$$

being the nonlinear second derivatives of the metric, is the relativistic tidal forces, its contracted version enters into the GR field equation (14.35)

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu} \tag{17.6}$$

where the Ricci tensor $R_{\mu\nu}$ and scalar R are contractions of the Riemann curvature

$$R_{\mu\nu} \equiv g^{\alpha\beta} R_{\alpha\mu\beta\nu} \quad \text{and} \quad R \equiv g^{\alpha\beta} R_{\alpha\beta} \tag{17.7}$$

and $T_{\mu\nu}$ is the energy–momentum tensor for an external source. κ is proportional to Newton's constant.

The Einstein–Hilbert action

Just as Maxwell's equation can be compactly presented as the Euler–Lagrangian equation resulting from the variation of the Maxwell action, the Einstein equation (17.6) is similarly related to the GR Lagrangian density, the Ricci scalar:

$$\mathcal{L}_g = R, \tag{17.8}$$

for the source-free case (Hilbert 1915, Einstein 1916d). Since only the product $\sqrt{-g} d^4x$ (where g is the determinant of the metric tensor $g_{\mu\nu}$) is invariant under the general coordinate transformation,[3] the relevant action, called the **Einstein–Hilbert action**, is the 4D integral

$$I_g = \int \sqrt{-g} d^4x \mathcal{L}_g = \int \sqrt{-g} d^4x g^{\mu\nu} R_{\mu\nu}. \tag{17.9}$$

We can then derive Eq. (17.6) as the Euler–Lagrange equation from the minimization of this action. The variation of the action δI_g has three parts involving $\delta R_{\mu\nu}$, $\delta g^{\mu\nu}$, and $\delta\sqrt{-g}$. The integral containing the $\delta R_{\mu\nu}$ factor after an integration by parts turns into a vanishing surface term; the metric matrix being symmetric[4] hence obeys the general relation $\ln(\det g_{\mu\nu}) = Tr(\ln g_{\mu\nu})$ leading

[3] For such more advanced GR topics, see, for example, Carroll (2004).

[4] A symmetric matrix M can always be diagonalized by a similarity transformation $SMS^\mathsf{T} = M_d$.

to the variation of $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$. Consequently the variation principle requires

$$\delta I_g = \int \sqrt{-g}d^4x\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\delta g^{\mu\nu} = 0,$$

which implies the Einstein equation of (17.6). In this sense we can interpret the general coordinate-invariant Ricci scalar R as representing the (source-free) 4D gravitational theory.

17.2 General relativity in 5D spacetime

In this section we show how a particular version of the 5D spacetime metric leads to the Ricci scalar being the sum of Lagrangian densities of Einstein's gravity theory and Maxwell's electromagnetism.

17.2.1 Extra spatial dimension and the Kaluza–Klein metric

One can motivate the geometric unification by the observation that while we have the 4-potential A_μ in electromagnetism, the spacetime metric $g_{\mu\nu}$ is the relativistic gravitational potential. One would like to combine these two types of potentials into one mathematical entity.[5]

Kaluza starts out by postulating a spacetime with an extra spatial dimension[6]

$$\hat{x}^M = \left(x^0, x^1, x^2, x^3, x^5\right). \tag{17.10}$$

However, the metric $\hat{g}_{MN}$ for this 5D spacetime is assumed to have a particular structure $\hat{g}_{MN} = \hat{g}^{(kk)}_{MN}$ having its elements related to the 4D $g_{\mu\nu}$ and A^μ as,

$$\hat{g}^{(kk)}_{\mu\nu} = g_{\mu\nu} + A_\mu A_\nu, \quad \hat{g}^{(kk)}_{\mu 5} = \hat{g}^{(kk)}_{5\mu} = A_\mu, \qquad \hat{g}^{(kk)}_{55} = 1. \tag{17.11}$$

When displayed in 5×5 matrix form, we have

$$\hat{g}^{(kk)}_{MN} \equiv \begin{pmatrix} \hat{g}^{(kk)}_{\mu\nu} & \hat{g}^{(kk)}_{\mu 5} \\ \hat{g}^{(kk)}_{5\nu} & \hat{g}^{(kk)}_{55} \end{pmatrix} = \begin{pmatrix} g_{\mu\nu} + A_\mu A_\nu & A_\mu \\ A_\nu & 1 \end{pmatrix}. \tag{17.12}$$

Equivalently, the corresponding invariant interval in this 5D spacetime can be written as

$$ds^2_{(kk)} = \hat{g}^{(kk)}_{MN} d\hat{x}^M d\hat{x}^N = g_{\mu\nu}dx^\mu dx^\nu + (dx^5 + A_\lambda dx^\lambda)^2. \tag{17.13}$$

We should note the particular feature that, with $g_{\mu\nu}$ and A^μ being functions of the 4D coordinate x^μ, all the elements of the 5D metric $\hat{g}^{(kk)}_{MN}$ have no dependence on the extra dimensional coordinate $\hat{x}^5$ and we also set $\hat{g}^{(kk)}_{55} = 1$. From this point on we shall drop the cumbersome superscript label (kk) and the relation $\hat{g}_{MN} = \hat{g}^{(kk)}_{MN}$ is always understood.

One can also check that the 5D inverse metric $\hat{g}^{MN}$ must have the components of

$$\hat{g}^{\mu\nu} = g^{\mu\nu}, \quad \hat{g}^{\mu 5} = \hat{g}^{5\mu} = -A^\mu, \qquad \hat{g}^{55} = 1 + A^\nu A_\nu, \tag{17.14}$$

[5] Further discussion can be found in Section 17.3.1.

[6] 5D quantities will be denoted with a caret symbol ^. The capital Latin index $M = (\mu, 5) = (0, 1, 2, 3, 5)$ is for a 5D spacetime, with the Greek index μ for the usual 4D spacetime and the index 5 for the extra spatial dimension. Our system skips the index 4, so as not to be confused with another common practice of labeling the 4D spacetime by the indices $(1, 2, 3, 4)$ with the fourth index being the time coordinate. In our system the time component continues to be denoted by the zeroth index.

or,

$$\hat{g}^{MN} = \begin{pmatrix} g^{\mu\nu} & -A^{\mu} \\ -A^{\nu} & 1 + A^{\lambda}A_{\lambda} \end{pmatrix} \tag{17.15}$$

so that a simple matrix multiplication can check out the correct metric relation $\hat{g}^{MN}\hat{g}_{MN} = \delta^M_K$.

17.2.2 "The Kaluza–Klein miracle"

From the 5D metric, we can obtain the other curved spacetime quantities by the usual relations. The **5D Christoffel symbols**, cf. Eq. (17.4), are first-order derivatives of the 5D metric

$$\hat{\Gamma}^M_{NL} = \frac{1}{2}\hat{g}^{MK}\left(\partial_N \hat{g}_{LK} + \partial_L \hat{g}_{NK} - \partial_K \hat{g}_{NL}\right). \tag{17.16}$$

The 5D **Riemann curvature tensor** is a nonlinear derivative of the 5D **Christoffel symbols**, cf. Eq. (17.5):

$$\hat{R}^L{}_{MSN} = \partial_S \hat{\Gamma}^L_{MN} - \partial_N \hat{\Gamma}^L_{MS} + \hat{\Gamma}^L_{ST}\hat{\Gamma}^T_{MN} - \hat{\Gamma}^L_{NT}\hat{\Gamma}^T_{MS}. \tag{17.17}$$

Contracting the pair of indices (L, S), we obtain the 5D **Ricci tensor**, cf. Eq. (14.23),

$$\hat{R}_{MN} = \hat{R}^L{}_{MLN} = \partial_L \hat{\Gamma}^L_{MN} - \partial_N \hat{\Gamma}^L_{ML} + \hat{\Gamma}^L_{LT}\hat{\Gamma}^T_{MN} - \hat{\Gamma}^L_{NT}\hat{\Gamma}^T_{ML}. \tag{17.18}$$

Contracting one more time, we obtain the **5D Ricci scalar**, cf. Eq. (14.25),

$$\hat{R} = \hat{g}^{MN}\hat{R}_{MN}. \tag{17.19}$$

Given Kaluza's stipulation of the 5D metric, it is a straightforward, but rather tedious, task to calculate the 5D Christoffel symbols, in terms of the familiar 4D metric tensor and 4D electromagnetic potential, and then all the other 5D geometric quantities as listed in Eqs. (17.16)–(17.19). After an enormous calculation,[7] Kaluza obtained the remarkable result that the 5D Ricci scalar $\hat{R}$, which should be the Lagrangian density of a 5D general theory of relativity, is simply the sum of the Lagrangian densities of the 4D general relativity R and Maxwell's electromagnetism, $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ where $F_{\mu\nu}$ is the Maxwell field tensor (17.2):

$$\hat{R} = R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{17.20}$$

namely,

$$\mathcal{L}^{(5)}_g = \mathcal{L}^{(4)}_g + \mathcal{L}^{(4)}_{\text{EM}}. \tag{17.21}$$

The Einstein and Maxwell equations are all components of the 5D GR field equation.[8] In this rather "miraculous" way a geometric unification of gravitation and electromagnetism is indicated.

[7] The details of calculating the 5D Ricci scalar $\hat{R}$ in term of $g_{\mu\nu}$ and A^{μ} are provided in SuppMat Section 17.5.

[8] It can be similarly shown that the 4D Lorentz force law follows from the 5D geodesic equation (the GR equation of motion).

17.3 The physics of the Kaluza–Klein spacetime

As indicated above, once we have the KK metric (17.11), the unification of gravitation and electromagnetism follows by a straightforward calculation. Thus the whole unification program relies on the structure of the KK metric. What is the physics behind the KK metric ansatz?

17.3.1 Motivating the Kaluza–Klein metric ansatz

The amazing unification results having their origin in the KK prescription (17.11) for the metric $\hat{g}^{(kk)}_{MN}$; it may be worthwhile to motivate the algebra that can lead one to this metric ansatz.

The metric $g_{\mu\nu}$ being the gravitational potential and comparable to the EM potential A_μ, one would like to fit both of them into the 5D $\hat{g}_{MN}$. What should be the precise identification of the metric elements? Similarly, can one fit the Christoffel symbols $\Gamma^{\mu}_{\nu\lambda}$ and EM field tensor $F_{\mu\nu}$ (both being first derivatives of the potentials) into $\hat{\Gamma}^{M}_{NL}$? Or, after lowering the upper index in (17.16),

$$\hat{\Gamma}_{M\,NL} \equiv \hat{g}_{MJ}\hat{\Gamma}^{J}_{NL} = \frac{1}{2}\left(\partial_N \hat{g}_{LM} + \partial_L \hat{g}_{NM} - \partial_M \hat{g}_{NL}\right). \tag{17.22}$$

Out of the 50 elements, we will concentrate on the set with the indices $M = \mu$, $L = 5$, and $N = \nu$ in (17.22) as a possible match for the EM field intensity $F_{\mu\nu}$:

$$\hat{\Gamma}_{\mu\,5\nu} = \frac{1}{2}\left(\partial_\nu \hat{g}_{5\mu} + \partial_5 \hat{g}_{\mu\nu} - \partial_\mu \hat{g}_{\nu 5}\right). \tag{17.23}$$

This suggests the identification with $-\frac{1}{2}F_{\mu\nu}$ if the 5D metric elements actually do not depend on the x^5 coordinate so that the middle term vanishes, $\partial_5 \hat{g}_{\mu\nu} = 0$, and if $\hat{g}_{\mu 5} = \hat{g}_{5\mu} = A_\mu$. With the further simplifying assumption of $\hat{g}_{55} = 1$ and ds^2, Kaluza ends up trying the ansatz of (17.11), hence (17.13).

17.3.2 Gauge transformation as a 5D coordinate change

The invariant interval $ds^2_{(kk)} = \hat{g}^{(kk)}_{MN} d\hat{x}^M d\hat{x}^N$, with the metric $\hat{g}^{(kk)}_{MN}$ not being the most general 5D metric tensor, will not be invariant under the most general coordinate transformation in the 5D spacetime $\hat{x}^M \to \hat{x}'^M$. However, $ds^2_{(kk)}$, as we shall show, is unchanged under a subset of coordinate transformations holding the 4D coordinates fixed and a local displacement of the extra dimensional coordinate:

$$x^\mu \to x'^\mu = x^\mu \quad \text{and} \quad x^5 \to x'^5 = x^5 + \theta(x). \tag{17.24}$$

Since we have $g_{\mu\nu}(x)$ depending on the 4D coordinate only, this leads to

$$g'_{\mu\nu}(x) = g_{\mu\nu}(x), \quad \text{and, of course,} \quad \hat{g}'_{55} = \hat{g}_{55} = 1. \tag{17.25}$$

Most interestingly, according to the general transformation rule, cf. Eq. (13.29), with $\partial \hat{x}^M / \partial x'^5 = \delta^M_5$, we have

$$\begin{aligned}
\hat{g}'_{5\mu} &= \frac{\partial \hat{x}^M}{\partial x'^5}\frac{\partial \hat{x}^N}{\partial x'^\mu}\hat{g}_{MN} = \frac{\partial \hat{x}^N}{\partial x'^\mu}\hat{g}_{5N} \\
&= \frac{\partial x^\nu}{\partial x'^\mu}\hat{g}_{5\nu} + \frac{\partial x^5}{\partial x'^\mu}\hat{g}_{55} = \delta^\nu_\mu \hat{g}_{5\nu} - \frac{\partial \theta}{\partial x^\mu}.
\end{aligned} \tag{17.26}$$

With the identification of $\hat{g}_{5\mu} = A_\mu$, this is just the gauge transformation of Eq. (17.1). It is then an easy exercise to check that the KK interval of (17.13) is unchanged under this restricted 5D coordinate transformation (17.25) and (17.26).

Recall the discussion in Chapter 16 and reviewed in Section 17.1 that electromagnetism can to a large extent be determined by the $U(1)$ gauge symmetry. As the $U(1)$ transformation $e^{i\theta(x)}$ is equivalent to a rotation by an angle θ, we can have the literal realization of the gauge transformation as a displacement along a circle if the x^5 coordinate is compactified into a circle. (See further discussion below.) That we can interpret gauge transformations as coordinate transformations in an extra dimension goes a long way in explaining why Maxwell's theory is embedded in this higher dimensional GR theory.

17.3.3 Compactified extra dimension

We have explained that the 5D coordinate transformation that leaves the KK interval invariant is a gauge transformation. Still, why does one restrict the metric to this $\hat{g}^{(kk)}_{MN}$ form?

The key feature of the KK metric is that its elements are independent of the extra dimensional coordinate x^5. This is rather strange: one postulates a spacetime with extra dimension $\left(x^0, x^1, x^2, x^3, x^5\right)$ yet the fields are not allowed to depend on the extra coordinate x^5!

Kaluza, the mathematician, is silent on the physical reality of the extra dimension; but Klein, the physicist, proposes that the fifth dimension is real. It has not been observed because it is extremely small; the extra dimension is curled up. Just like a garden hose is viewed at a distance as a 1D line, upon closer inspection one finds a surface composed of a series of circles (Fig. 17.1). Here one of the two dimensions of the surface is compactified into circles. In the same manner, Klein proposes that one spatial dimension of our 5D spacetime is compactified: every point in the observed 3D spatial space is actually a circle.

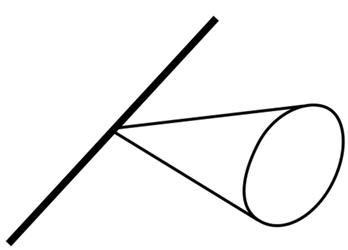

Fig. 17.1 Compactified dimension. A 1D line is revealed to be a 2D surface with one dimension compactified into a circle.

17.3.4 Quantum fields in a compactified space

The Kaluza–Klein "miracle" was discovered by Kaluza, and was explained by Klein, who used the then new quantum mechanics to deduce the consequence of a compactified dimension. This also justifies the restrictions imposed on the 5D metric (Klein 1926).

Consider, as the simplest case, a scalar field $\phi\left(\hat{x}^M\right) = \phi\left(x^\mu, x^5\right)$ satisfying a 5D relativistic wave equation (the Klein–Gordon equation):

$$\left(\Box^{(5)} - \frac{m_0^2 c^2}{\hbar^2}\right)\phi\left(\hat{x}^N\right) = 0 \tag{17.27}$$

where $\Box^{(5)}$ is the five-dimensional D'Alembertian operator[9] $\Box^{(5)} = \Box^{(4)} + \partial^2/\partial x_5^2$. Since the extra dimension is a circle (with compactification radius a), we have the identification of x^5 and $x^5 + 2\pi a$. Thus the wavefunction $\phi\left(\hat{x}^M\right)$ must satisfying the boundary condition of

[9]Cf. Eq. (11.25).

$$\phi\left(x^{\mu},x^{5}\right)=\phi\left(x^{\mu},x^{5}+2\pi a\right). \tag{17.28}$$

The field must have a sinusoidal dependence on the x^5 coordinate, and has the harmonic expansion

$$\phi\left(x^{\mu},x^{5}\right)=\sum_{n}\phi_{n}\left(x^{\mu}\right)e^{ip_{n}x^{5}/\hbar}. \tag{17.29}$$

In order that the boundary condition (17.28) is satisfied, the momentum in the extra dimension must be quantized[10]

$$p_{n}=n\frac{\hbar}{a}\quad\text{with}\quad n=0,1,2,\ldots \tag{17.30}$$

[10]Recall the similar problem of "a particle in a box" in quantum mechanics.

A tower of Kaluza–Klein particles in 4D spacetime

To see the implications in the familiar 4D spacetime, we can write out the 5D Klein–Gordon equation in the 4D spacetime

$$\left(\Box^{(5)}-\frac{m_{0}^{2}c^{2}}{\hbar^{2}}\right)\phi\left(\hat{x}^{N}\right)=\left(\Box^{(4)}+\partial_{5}^{2}-\frac{m_{0}^{2}c^{2}}{\hbar^{2}}\right)\phi\left(\hat{x}^{N}\right)=0,$$

which, after substituting in the series expansion, becomes

$$\sum_{n}\left[\left(\Box^{(4)}-\frac{n^{2}}{a^{2}}-\frac{m_{0}^{2}c^{2}}{\hbar^{2}}\right)\phi\left(x^{\mu}\right)\right]e^{ip_{n}x^{5}/\hbar.}=0.$$

Namely, we have an infinite number of decoupled 4D Klein–Gordon equations

$$\left(\Box^{(4)}-\frac{m_{n}^{2}c^{2}}{\hbar^{2}}\right)\phi\left(x^{\mu}\right)=0, \tag{17.31}$$

with a tower of "Kaluza–Klein states" having masses

$$m_{n}^{2}=m_{0}^{2}+n^{2}\frac{\hbar^{2}}{a^{2}c^{2}}. \tag{17.32}$$

Thus the signature of the extra dimension in 4D spacetime is a tower of KK particles, having identical spin and gauge quantum numbers, with increasing masses controlled by the compactification scale of a.

Compactification by quantum gravity?

The natural expectation is that the compactification is brought about by the dynamics of quantum gravity. In this way the compactification radius should be the order of the Planck length, discussed in Section 3.3.2. That the first KK state has a mass of at least 10^{19} GeV would mean that such particles would not be detectable in the foreseeable future. Nevertheless, the "decoupling" of the large KK state masses do explain the basic structure of the KK metric ansatz—the x^5 independence of the metric elements.

Since only the $n=0$ state is physically relevant, we have from Eqs. (17.30) and (17.29) the approximation

$$\phi\left(x^{\mu},x^{5}\right)=\phi_{0}\left(x^{\mu}\right) \tag{17.33}$$

and the x^5 dependence of the theory disappears.

17.4 Further theoretical developments

We have presented the Kaluza–Klein theory as an illustration of Einstein's unified field theory. While it is an example of unification of fundamental forces, it certainly does not have a bearing on Einstein's loftier goal of a unified theory that would explain the mystery of quantum physics. In fact KK theory makes use of conventional quantum mechanics in extracting physical consequences of a compactified spatial dimension.

Extending the original Kaluza–Klein theory

Scalar–tensor gravity theory Even restricted to the original theory, there is no strong theoretical argument for setting the metric element $g_{55} = 1$. A more natural alternative is to replace it by a field. This would lead to a scalar–tensor theory of gravity, for which there is no experimental support.

From a circle to Calabi–Yau space From a modern perspective, the KK theory cannot be a complete unified theory because the list of fundamental forces must be expanded beyond the gravitational and electromagnetic interactions: it must at least include the strong and weak particle interactions.[11] Nevertheless, the modern development of particle physics has led to the discovery of superstring theory as a possible quantum gravity theory that has the potential to unify all fundamental forces. What is most relevant for our discussion here is the finding that the self-consistency requirement of superstring theory requires a spacetime to have 10 dimensions. Thus what is needed is not just one extra dimension curled into a very small circle but six extra dimensions into a more complicated geometric entity. A much discussed compactification scheme is the Calabi–Yau space.[12] In short, the spirit of Einstein's unification program, especially in the form of Kaluza–Klein extra dimensions, is being carried on in the foremost theoretical physics research of the twenty-first century.

[11] In this connection, we note that Oskar Klein in the late 1930s constructed a 5D theory that attempted to include not only gravity and electromagnetism, but also Yukawa-meson-mediated nuclear forces. Although he did not explicitly consider any nonabelian gauge symmetry, his theory had foreshadowed the later development of Yang–Mills theory, in particular, charged gauge bosons, etc. He presented these results in a 1938 conference held in Warsaw, but never published them formally. For an appreciation of this 1938 contribution see Gross (1995).

[12] For a comprehensive and nontechnical discussion, see Yau and Nadis (2010).

Speculations of a large extra dimension As the current thinking of unification theory is directly related to the quest for quantum gravity, the natural unification distance scale is thought to be the Planck length, which is something like nine orders of magnitude smaller than the highest accelerator can probe. This makes an experimental test of such theories extremely difficult. Yet researchers have found the whole idea of extra dimensions so attractive that there have been serious speculations on the possibility that the compactification scale is much larger than the Planck size. Maybe the extra dimension is on the electroweak scale that can possibly be revealed in experiments being performed at the Large Hadron Collider. These are intriguing speculations that are being actively pursued.

17.4.1 Lessons from Maxwell's equations

In this book we have repeatedly discussed the importance of Maxwell's equations. We will conclude our presentation by recalling the important lessons that we have learnt from the structure of these equations:

- The idea of the photon and quantum theory, brought forth through a deep statistical thermodynamic study of electromagnetic radiation.
- Einstein's principle of relativity, that taught us that the arena of physics is 4D spacetime; this paved the way for a geometric understanding of gravitation.
- The idea of local symmetry in the charge space, leading eventually to the viewpoint that all fundamental interactions have a connection with gauge symmetry.
- Finally, the possibility that spacetime has extra dimensions. This may be the origin of the charge space.

17.4.2 Einstein and mathematics

"Mathematics is the language of physics." Such a statement implies a rather passive role for mathematics. In fact this language has often led the way in opening up new understanding in physics. This is one of the powerful lessons that one gathers from the history of theoretical physics; this account of Einstein's physics, we believe, confirms this opinion too. Clearly Einstein's discovery that Riemannian geometry offers a truer description of nature is a brilliant example of such a role. One aspect of Einstein's scientific biography can be viewed as the story of his growing appreciation of the role of mathematics: starting from his skepticism of higher mathematics, doubting the usefulness of Minkowski's geometric formulation of special relativity to the role that Riemannian geometry played in the implementation of the general principle of relativity—first learning the new mathematics with the help of Marcel Grossmann and the eventual discovery of the GR field equation after much struggle on his own. In the process, Einstein became greatly appreciative of the role of mathematics as fundamental in setting up new physical theory. Finding the correct mathematical structure to describe the physical concepts, and postulating the simplest equation compatible with that structure, are all key elements in the invention of new theories.

That Einstein had not made more advances in his unified field theory program may also imply that this great physicist was after all not a comparably great mathematician (like the case of Newton). One probably needs to invent new mathematics in order to forge progress in such a pursuit.[13] It is also interesting to observe the important contributions that mathematicians have made in furthering Einstein's vision: Hermann Weyl's gauge symmetry program and Theodor Kaluza's 5D GR leading to the 'Kaluza–Klein miracle'.

[13]In this connection see the comments made by Roger Penrose in the new *Forward* he wrote in the 2005 re-issue of Pais (1982).

17.5 SuppMat: Calculating the 5D tensors

Here we provide the details of calculating the 5D tensors in terms of the gravitational potential $g_{\mu\nu}$ and electromagnetic potential A_μ—all based on the KK metric prescription given by (17.11).

17.5.1 The 5D Christoffel symbols

The 5D Christoffel symbols $\hat{\Gamma}^{M}_{NL}$ are first-order derivatives of the 5D metric (17.16). They have six distinctive types of terms:

$$\hat{\Gamma}^{\mu}_{\nu\lambda},\ \hat{\Gamma}^{5}_{\nu\lambda},\ \hat{\Gamma}^{\mu}_{5\lambda},\ \hat{\Gamma}^{5}_{5\lambda},\ \hat{\Gamma}^{\mu}_{55},\ \hat{\Gamma}^{5}_{55}.$$

We shall calculate them in terms of the 4D metric $g_{\mu\nu}$, the electromagnetic potential A^{μ}, and the field tensor $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$.

The components $\hat{\Gamma}^{\mu}_{\nu\lambda}$

According to (17.16), with $M = \mu, N = \nu, L = \lambda$, we have

$$\begin{aligned}\hat{\Gamma}^{\mu}_{\nu\lambda} &= \frac{1}{2}\hat{g}^{\mu K}\left(\partial_{\nu}\hat{g}_{\lambda K} + \partial_{\lambda}\hat{g}_{\nu K} - \partial_{K}\hat{g}_{\nu\lambda}\right)\\ &= \frac{1}{2}\hat{g}^{\mu\rho}\left(\partial_{\nu}\hat{g}_{\lambda\rho} + \partial_{\lambda}\hat{g}_{\nu\rho} - \partial_{\rho}\hat{g}_{\nu\lambda}\right)\\ &\quad + \frac{1}{2}\hat{g}^{\mu 5}\left(\partial_{\nu}\hat{g}_{\lambda 5} + \partial_{\lambda}\hat{g}_{\nu 5} - \partial_{5}\hat{g}_{\nu\lambda}\right).\end{aligned} \tag{17.34}$$

Plugging in the Kaluza metric components of (17.11) and (17.14), we find the first term on the RHS is just the 4D Christoffel symbols $\Gamma^{\mu}_{\nu\lambda}$ with an extra term coming from that fact that $\hat{g}_{\mu\nu} - g_{\mu\nu} = A_{\mu}A_{\nu}$, while in the second term we have $\partial_5\hat{g}_{\nu\lambda} = 0$ because the Kaluza metric elements are independent of the extra coordinate x_5.

$$\begin{aligned}\hat{\Gamma}^{\mu}_{\nu\lambda} = \Gamma^{\mu}_{\nu\lambda} &+ \frac{1}{2}g^{\mu\rho}\left[\partial_{\nu}\left(A_{\lambda}A_{\rho}\right) + \partial_{\lambda}\left(A_{\nu}A_{\rho}\right) - \partial_{\rho}\left(A_{\nu}A_{\lambda}\right)\right]\\ &- \frac{1}{2}A^{\mu}\left(\partial_{\nu}A_{\lambda} + \partial_{\lambda}A_{\nu}\right).\end{aligned} \tag{17.35}$$

The first two terms in the square bracket on the RHS can be written out as

$$\begin{aligned}&\frac{1}{2}g^{\mu\rho}\left[\partial_{\nu}\left(A_{\lambda}A_{\rho}\right) + \partial_{\lambda}\left(A_{\nu}A_{\rho}\right)\right]\\ &= \frac{1}{2}g^{\mu\rho}\left[\left(\partial_{\nu}A_{\lambda}\right)A_{\rho} + A_{\lambda}\left(\partial_{\nu}A_{\rho}\right) + \left(\partial_{\lambda}A_{\nu}\right)A_{\rho} + A_{\nu}\left(\partial_{\lambda}A_{\rho}\right)\right]\\ &= \frac{1}{2}\left[\left(\partial_{\nu}A_{\lambda}\right)A^{\mu} + g^{\mu\rho}A_{\lambda}\left(\partial_{\nu}A_{\rho}\right) + \left(\partial_{\lambda}A_{\nu}\right)A^{\mu} + g^{\mu\rho}A_{\nu}\left(\partial_{\lambda}A_{\rho}\right)\right].\end{aligned} \tag{17.36}$$

The first and third terms in this last square bracket just cancel the last term on the RHS of (17.35). On the other hand the last term in the square bracket on the RHS of (17.35), when expanded,

$$-\frac{1}{2}g^{\mu\rho}\partial_{\rho}\left(A_{\nu}A_{\lambda}\right) = -\frac{1}{2}g^{\mu\rho}\left[\left(\partial_{\rho}A_{\nu}\right)A_{\lambda} + A_{\nu}\left(\partial_{\rho}A_{\lambda}\right)\right] \tag{17.37}$$

can be combined with the second and fourth terms on the RHS of (17.36) to yield

$$\frac{1}{2}g^{\mu\rho}\left[\left(\partial_{\nu}A_{\rho} - \partial_{\rho}A_{\nu}\right)A_{\lambda} + \left(\partial_{\lambda}A_{\rho} - \partial_{\rho}A_{\lambda}\right)A_{\nu}\right] = -\frac{1}{2}g^{\mu\rho}\left(F_{\rho\nu}A_{\lambda} + F_{\rho\lambda}A_{\nu}\right).$$

All this leads to

$$\hat{\Gamma}^{\mu}_{\nu\lambda} = \Gamma^{\mu}_{\nu\lambda} - \frac{1}{2} g^{\mu\rho} \big(F_{\rho\nu} A_{\lambda} + F_{\rho\lambda} A_{\nu} \big). \tag{17.38}$$

Later on we shall also need the Christoffel symbols $\hat{\Gamma}^{\mu}_{\nu\lambda}$ with a pair of indices summed over:

$$\hat{\Gamma}^{\mu}_{\nu\mu} = \Gamma^{\mu}_{\nu\mu} - \frac{1}{2} g^{\mu\rho} \big(F_{\rho\nu} A_{\mu} + F_{\mu\rho} A_{\nu} \big).$$

The very last factor vanishes because of the opposite symmetry properties of the two tensors: $g^{\mu\rho} F_{\mu\rho} = 0$. Expanding out $\Gamma^{\mu}_{\nu\mu}$, we then have

$$\hat{\Gamma}^{\mu}_{\nu\mu} = \frac{1}{2} g^{\mu\rho} \big(\partial_{\nu} g_{\mu\rho} + \partial_{\mu} g_{\nu\rho} - \partial_{\rho} g_{\nu\mu} - F_{\rho\nu} A_{\mu} \big).$$

The two middle terms cancel, $g^{\mu\rho} \big(\partial_{\mu} g_{\nu\rho} - \partial_{\rho} g_{\nu\mu} \big) = 0$, and this leads to

$$\hat{\Gamma}^{\mu}_{\nu\mu} = \frac{1}{2} g^{\mu\rho} \partial_{\nu} g_{\mu\rho} - \frac{1}{2} A_{\mu} F^{\mu}_{\nu}. \tag{17.39}$$

The components $\hat{\Gamma}^{5}_{\nu\lambda}$

According to (17.16), with $M = 5, N = \nu, L = \lambda$,

$$\begin{aligned}
\hat{\Gamma}^{5}_{\nu\lambda} &= \frac{1}{2} \hat{g}^{5K} \big(\partial_{\nu} \hat{g}_{\lambda K} + \partial_{\lambda} \hat{g}_{\nu K} - \partial_{K} \hat{g}_{\nu\lambda} \big) \\
&= \frac{1}{2} \hat{g}^{5\rho} \big(\partial_{\nu} \hat{g}_{\lambda\rho} + \partial_{\lambda} \hat{g}_{\nu\rho} - \partial_{\rho} \hat{g}_{\nu\lambda} \big) \\
&\quad + \frac{1}{2} \hat{g}^{55} \big(\partial_{\nu} \hat{g}_{\lambda 5} + \partial_{\lambda} \hat{g}_{\nu 5} - \partial_{5} \hat{g}_{\nu\lambda} \big).
\end{aligned} \tag{17.40}$$

Plugging in the Kaluza metric components of (17.11) and (17.14), and noting $\partial_5 \hat{g}_{\nu\lambda} = 0$, we have

$$\begin{aligned}
\hat{\Gamma}^{5}_{\nu\lambda} = {}& -A^{\rho} \Gamma_{\rho\nu\lambda} - \frac{1}{2} A^{\rho} \left[\partial_{\nu} \big(A_{\lambda} A_{\rho} \big) + \partial_{\lambda} \big(A_{\nu} A_{\rho} \big) - \partial_{\rho} \big(A_{\lambda} A_{\nu} \big) \right] \\
& + \frac{1}{2} \big(1 + A^{\sigma} A_{\sigma} \big) \big(\partial_{\nu} A_{\lambda} + \partial_{\lambda} A_{\nu} \big).
\end{aligned} \tag{17.41}$$

The first two terms in the square bracket can be written out as

$$\begin{aligned}
& -\frac{1}{2} A^{\rho} \left[\partial_{\nu} \big(A_{\lambda} A_{\rho} \big) + \partial_{\lambda} \big(A_{\nu} A_{\rho} \big) \right] \\
= {}& \frac{1}{2} \left[-A^{\rho} A_{\rho} \big(\partial_{\nu} A_{\lambda} \big) - A^{\rho} \big(\partial_{\nu} A_{\rho} \big) A_{\lambda} - A^{\rho} A_{\rho} \big(\partial_{\lambda} A_{\nu} \big) - A^{\rho} \big(\partial_{\lambda} A_{\rho} \big) A_{\nu} \right].
\end{aligned}$$

We note that the first and third terms on the RHS just cancel the last two terms on the RHS of (17.41). The remaining second and fourth terms combine with the last term in the square bracket of (17.41)

$$-\frac{1}{2}A^\rho\left[\left(\partial_\nu A_\rho\right)A_\lambda + \left(\partial_\lambda A_\rho\right)A_\nu - \partial_\rho\left(A_\lambda A_\nu\right)\right]$$
$$-\frac{1}{2}A^\rho\left[\left(\partial_\nu A_\rho\right)A_\lambda - \left(\partial_\rho A_\nu\right)A_\lambda + \left(\partial_\lambda A_\rho\right)A_\nu - \left(\partial_\rho A_\lambda\right)A_\nu\right]$$
$$= -\frac{1}{2}A^\rho\left[F_{\nu\rho}A_\lambda + F_{\lambda\rho}A_\nu\right]. \tag{17.42}$$

Equation (17.41) then becomes

$$\hat{\Gamma}^5_{\nu\lambda} = -A^\rho\Gamma_{\rho\nu\lambda} - \frac{1}{2}A^\rho\left[F_{\nu\rho}A_\lambda + F_{\lambda\rho}A_\nu\right] + \frac{1}{2}B_{\nu\lambda} \tag{17.43}$$

where

$$B_{\nu\lambda} = \partial_\nu A_\lambda + \partial_\lambda A_\nu. \tag{17.44}$$

The components $\hat{\Gamma}^\mu_{5\lambda}$

According to (17.16), with $M = \mu, N = 5, L = \lambda$,

$$\hat{\Gamma}^\mu_{5\lambda} = \frac{1}{2}\hat{g}^{\mu K}\left(\partial_5\hat{g}_{\lambda K} + \partial_\lambda\hat{g}_{5K} - \partial_K\hat{g}_{5\lambda}\right). \tag{17.45}$$

With $\partial_5\hat{g}_{\lambda K} = 0$ and separating the K-index summation into a $K = \rho$ sum and a $K = 5$ sum:

$$\hat{\Gamma}^\mu_{5\lambda} = \frac{1}{2}\hat{g}^{\mu\rho}\left(\partial_\lambda\hat{g}_{5\rho} - \partial_\rho\hat{g}_{5\lambda}\right) + \frac{1}{2}\hat{g}^{\mu 5}\left(\partial_\lambda\hat{g}_{55} - \partial_5\hat{g}_{5\lambda}\right). \tag{17.46}$$

Again we have $\partial_5\hat{g}_{5\lambda} = 0$ and as $\hat{g}_{55} = 1$ so that $\partial_\lambda\hat{g}_{55} = 0$;

$$\hat{\Gamma}^\mu_{5\lambda} = \frac{1}{2}g^{\mu\rho}\left(\partial_\lambda A_\rho - \partial_\rho A_\lambda\right) = \frac{1}{2}g^{\mu\rho}F_{\lambda\rho} = -\frac{1}{2}F^\mu_\lambda. \tag{17.47}$$

Recall from Section 17.3.1 that it is this simple relation that has motivated the original ansatz for the KK metric. Furthermore we note that the sum over the indices μ and λ in $\hat{\Gamma}^\mu_{5\lambda}$ vanishes because $g^{\mu\rho}$ is symmetric and $F_{\mu\rho}$ is antisymmetric:

$$\hat{\Gamma}^\mu_{5\mu} = \frac{1}{2}g^{\mu\rho}F_{\mu\rho} = 0. \tag{17.48}$$

The components $\hat{\Gamma}^5_{5\lambda}$

Following the same steps leading to (17.47), we now calculate the $M = 5$, $L = \lambda$, and $\mu = 5$ element:

$$\hat{\Gamma}^5_{5\lambda} = \frac{1}{2}g^{5\rho}\left(\partial_\lambda A_\rho - \partial_\rho A_\lambda\right) = -\frac{1}{2}F_{\lambda\rho}A^\rho. \tag{17.49}$$

The components $\hat{\Gamma}^\mu_{55}$ and $\hat{\Gamma}^5_{55}$

According to (17.46), with $\lambda = 5$, we have

$$\hat{\Gamma}^\mu_{55} = \frac{1}{2}\hat{g}^{\mu\rho}\left(\partial_5\hat{g}_{5\rho} - \partial_\rho\hat{g}_{55}\right) = 0 \tag{17.50}$$

for the same reason that the last factor in (17.46) vanishes.

Similarly, with $\lambda = 5$, Eq. (17.49) becomes

$$\hat{\Gamma}^5_{55} = \frac{1}{2} g^{5\rho}\left(\partial_5 A_\rho - \partial_\rho \hat{g}_{55}\right) = 0. \tag{17.51}$$

Collecting all the $\hat{\Gamma}^M_{NL}$ components in one place, we have from (17.38), (17.43), (17.47), (17.49), (17.50), and (17.51):

$$\begin{aligned}
\hat{\Gamma}^\mu_{\nu\lambda} &= \Gamma^\mu_{\nu\lambda} - \frac{1}{2}\left(F^\mu_\nu A_\lambda + F^\mu_\lambda A_\nu\right) \\
\hat{\Gamma}^5_{\nu\lambda} &= -A_\rho \Gamma^\rho_{\nu\lambda} - \frac{1}{2} A^\rho \left(F_{\nu\rho} A_\lambda + F_{\lambda\rho} A_\nu\right) + \frac{1}{2} B_{\nu\lambda} \\
\hat{\Gamma}^\mu_{5\lambda} &= -\frac{1}{2} F^\mu_\lambda \\
\hat{\Gamma}^5_{5\lambda} &= -\frac{1}{2} F_{\lambda\rho} A^\rho \\
\hat{\Gamma}^\mu_{55} &= \hat{\Gamma}^5_{55} = 0.
\end{aligned} \tag{17.52}$$

17.5.2 The 5D Ricci tensor components

Knowing the Christoffel symbols, we are ready to calculate the Ricci tensor $\hat{R}_{MN}$ according to Eq (17.18).

The 5D Ricci tensor components $\hat{R}_{\mu\nu}$

There are two pairs of repeated indices, L and T in (17.18). We will now consider the separate cases when they take on 4D values of $L = \lambda$ and $T = \tau$, or the extra dimensional value of 5.

1. $L = \lambda$ and $T = \tau$:

$$\left(\hat{R}_{\mu\nu}\right)_1 = \underbrace{\partial_\lambda \hat{\Gamma}^\lambda_{\mu\nu}}_{(1)} \underbrace{- \partial_\nu \hat{\Gamma}^\lambda_{\mu\lambda}}_{(2)} + \underbrace{\hat{\Gamma}^\lambda_{\lambda\tau} \hat{\Gamma}^\tau_{\mu\nu}}_{(3)} \underbrace{- \hat{\Gamma}^\lambda_{\nu\tau} \hat{\Gamma}^\tau_{\mu\lambda}}_{(4)}. \tag{17.53}$$

2. $L = 5$ and $T = \tau$:

$$\left(\hat{R}_{\mu\nu}\right)_2 = \partial_5 \hat{\Gamma}^5_{\mu\nu} \underbrace{- \partial_\nu \hat{\Gamma}^5_{\mu 5}}_{(5)} + \underbrace{\hat{\Gamma}^5_{5\tau} \hat{\Gamma}^\tau_{\mu\nu}}_{(6)} \underbrace{- \hat{\Gamma}^5_{\nu\tau} \hat{\Gamma}^\tau_{\mu 5}}_{(7)}. \tag{17.54}$$

3. $L = \lambda$ and $T = 5$:

$$\left(\hat{R}_{\mu\nu}\right)_3 = \hat{\Gamma}^\lambda_{\lambda 5} \hat{\Gamma}^5_{\mu\nu} \underbrace{- \hat{\Gamma}^\lambda_{\nu 5} \hat{\Gamma}^5_{\mu\lambda}}_{(7)}. \tag{17.55}$$

4. $L = T = 5$:

$$\left(\hat{R}_{\mu\nu}\right)_4 = \hat{\Gamma}^5_{55} \hat{\Gamma}^5_{\mu\nu} \underbrace{- \hat{\Gamma}^5_{\nu 5} \hat{\Gamma}^5_{\mu 5}}_{(8)}. \tag{17.56}$$

Out of the 12 terms on the RHS, three terms vanish: besides the absence of the x^5 dependence $\partial_5 \hat{\Gamma}^5_{\mu\nu} = \hat{\Gamma}^5_{55} \hat{\Gamma}^5_{\mu\nu} = 0$, we also have $\hat{\Gamma}^\lambda_{\lambda 5} \hat{\Gamma}^5_{\mu\nu} = 0$ because, according to (17.52), $\hat{\Gamma}^{\lambda}_{\lambda 5} = -\frac{1}{2}F^{\lambda}_{\lambda} = 0$. Furthermore, the last term in (17.55) may be written as

$$-\hat{\Gamma}^{\lambda}_{\nu 5}\hat{\Gamma}^{5}_{\mu\lambda} = -\hat{\Gamma}^{5}_{\mu\lambda}\hat{\Gamma}^{\lambda}_{\nu 5} = -\hat{\Gamma}^{5}_{\mu\tau}\hat{\Gamma}^{\tau}_{\nu 5}$$

where we reach the last expression by changing the labels of the dummy indices from λ to τ. This makes it clear that this term is really the same as the last term in (17.54); they are symmetrized with respect to the indices (μ, ν) which is justified because the Ricci tensor $\hat{R}_{\mu\nu}$ must be symmetric. As a result, altogether we have eight distinctive terms—four from (17.53), three from (17.54), and one from (17.56).

In the following we shall substitute into these eight terms the expression of $\hat{\Gamma}$ as given in (17.52). Keep in mind that, in the first four terms, term 1 to term 4, the 4D Γs in the 5D $\hat{\Gamma}$ just combine to form the 4D Ricci tensor $R_{\mu\nu}$. For the remaining terms, we have

(1) The $\partial_\lambda \hat{\Gamma}^{\lambda}_{\mu\nu}$ term:

$$\begin{aligned}\partial_\lambda \hat{\Gamma}^{\lambda}_{\mu\nu} &= -\frac{1}{2}\partial_\lambda\left(F^{\lambda}_{\mu}A_\nu + F^{\lambda}_{\nu}A_\mu\right)\\ &= -\frac{1}{2}\left(A_\nu\partial_\lambda F^{\lambda}_{\mu} + A_\mu\partial_\lambda F^{\lambda}_{\nu} + F^{\lambda}_{\mu}\partial_\lambda A_\nu + F^{\lambda}_{\nu}\partial_\lambda A_\mu\right). \qquad (17.57)\end{aligned}$$

(2) The $-\partial_\nu \hat{\Gamma}^{\lambda}_{\mu\lambda}$ term:

$$-\partial_\nu \hat{\Gamma}^{\lambda}_{\mu\lambda} = \frac{1}{2}\partial_\nu\left(F^{\lambda}_{\mu}A_\lambda + F^{\lambda}_{\lambda}A_\mu\right) = \frac{1}{2}\left(A_\lambda\partial_\nu F^{\lambda}_{\mu} + F^{\lambda}_{\mu}\partial_\nu A_\lambda\right). \qquad (17.58)$$

(3) The $\hat{\Gamma}^{\lambda}_{\lambda\tau}\hat{\Gamma}^{\tau}_{\mu\nu}$ term:

$$\begin{aligned}\hat{\Gamma}^{\lambda}_{\lambda\tau}\hat{\Gamma}^{\tau}_{\mu\nu} &= -\frac{1}{2}\left(F^{\lambda}_{\lambda}A_\tau + F^{\lambda}_{\tau}A_\lambda\right)\Gamma^{\tau}_{\mu\nu} - \frac{1}{2}\Gamma^{\lambda}_{\lambda\tau}\left(F^{\tau}_{\mu}A_\nu + F^{\tau}_{\nu}A_\mu\right)\\ &\quad + \frac{1}{4}\left(F^{\lambda}_{\lambda}A_\tau + F^{\lambda}_{\tau}A_\lambda\right)\left(F^{\tau}_{\mu}A_\nu + F^{\tau}_{\nu}A_\mu\right)\\ &= -\frac{1}{2}A_\lambda\Gamma^{\tau}_{\mu\nu}F^{\lambda}_{\tau} - \frac{1}{2}A_\nu\Gamma^{\lambda}_{\lambda\tau}F^{\tau}_{\mu} - \frac{1}{2}A_\mu\Gamma^{\lambda}_{\lambda\tau}F^{\tau}_{\nu}\\ &\quad + \frac{1}{4}F^{\lambda}_{\tau}F^{\tau}_{\mu}A_\lambda A_\nu + \frac{1}{4}F^{\lambda}_{\tau}F^{\tau}_{\nu}A_\lambda A_\mu. \qquad (17.59)\end{aligned}$$

(4) The $-\hat{\Gamma}^{\lambda}_{\nu\tau}\hat{\Gamma}^{\tau}_{\mu\lambda}$ term:

$$\begin{aligned}-\hat{\Gamma}^{\lambda}_{\nu\tau}\hat{\Gamma}^{\tau}_{\mu\lambda} &= \frac{1}{2}\left(F^{\lambda}_{\nu}A_\tau + F^{\lambda}_{\tau}A_\nu\right)\Gamma^{\tau}_{\mu\lambda} + \frac{1}{2}\Gamma^{\lambda}_{\nu\tau}\left(F^{\tau}_{\mu}A_\lambda + F^{\tau}_{\lambda}A_\mu\right)\\ &\quad - \frac{1}{4}\left(F^{\lambda}_{\nu}A_\tau + F^{\lambda}_{\tau}A_\nu\right)\left(F^{\tau}_{\mu}A_\lambda + F^{\tau}_{\lambda}A_\mu\right)\\ &= \frac{1}{2}\left(A_\tau\Gamma^{\tau}_{\mu\lambda}F^{\lambda}_{\nu} + A_\nu\Gamma^{\tau}_{\mu\lambda}F^{\lambda}_{\tau} + A_\lambda\Gamma^{\lambda}_{\nu\tau}F^{\tau}_{\mu} + A_\mu\Gamma^{\lambda}_{\nu\tau}F^{\tau}_{\lambda}\right)\\ &\quad - \frac{1}{4}\left(F^{\lambda}_{\nu}F^{\tau}_{\mu}A_\tau A_\lambda + F^{\lambda}_{\tau}F^{\tau}_{\mu}A_\nu A_\lambda + F^{\lambda}_{\tau}F^{\tau}_{\lambda}A_\nu A_\mu + F^{\lambda}_{\nu}F^{\tau}_{\lambda}A_\tau A_\mu\right).\end{aligned} \qquad (17.60)$$

(5) The $-\partial_\nu \hat{\Gamma}^5_{\mu 5}$ term:

$$-\partial_\nu \hat{\Gamma}^5_{\mu 5} = \frac{1}{2}\partial_\nu \left(F_{\mu\tau}A^\tau\right) = \frac{1}{2}A^\tau \partial_\nu F_{\mu\tau} + \frac{1}{2}F_{\mu\tau}\partial_\nu A^\tau. \qquad (17.61)$$

(6) The $\hat{\Gamma}^5_{5\tau}\hat{\Gamma}^\tau_{\mu\nu}$ term:

$$\begin{aligned}\hat{\Gamma}^5_{5\tau}\hat{\Gamma}^\tau_{\mu\nu} &= -\frac{1}{2}F_{\tau\rho}A^\rho \Gamma^\tau_{\mu\nu} + \frac{1}{4}F_{\tau\rho}A^\rho \left(F^\tau_\mu A_\nu + F^\tau_\nu A_\mu\right) \\ &= -\frac{1}{2}A^\lambda \Gamma^\tau_{\mu\nu}F_{\tau\lambda} - \frac{1}{4}F_{\lambda\tau}F^\tau_\mu A^\lambda A_\nu - \frac{1}{4}F_{\lambda\tau}F^\tau_\nu A^\lambda A_\mu. \qquad (17.62)\end{aligned}$$

(7) The $-2\hat{\Gamma}^5_{\nu\tau}\hat{\Gamma}^\tau_{\mu 5}$ term:

$$\begin{aligned}-2\hat{\Gamma}^5_{\nu\tau}\hat{\Gamma}^\tau_{\mu 5} &= -A_\rho \Gamma^\rho_{\nu\tau}F^\tau_\mu - \frac{1}{2}A^\rho\left(F_{\nu\rho}A_\tau + F_{\tau\rho}A_\nu\right)F^\tau_\mu + \frac{1}{2}B_{\nu\tau}F^\tau_\mu \\ &= -A_\rho \Gamma^\rho_{\nu\tau}F^\tau_\mu - \frac{1}{2}F_{\nu\rho}F^\tau_\mu A^\rho A_\tau \\ &\quad -\frac{1}{2}F_{\tau\rho}F^\tau_\mu A^\rho A_\nu + \frac{1}{2}F^\tau_\mu B_{\nu\tau}. \qquad (17.63)\end{aligned}$$

(8) The $-\hat{\Gamma}^5_{\nu 5}\hat{\Gamma}^5_{\mu 5}$ term:

$$-\hat{\Gamma}^5_{\nu 5}\hat{\Gamma}^5_{\mu 5} = -\frac{1}{4}F_{\nu\rho}F_{\mu\lambda}A^\rho A^\lambda. \qquad (17.64)$$

Collecting all terms of the type $F\partial A$: the last two terms in (17.57), one from (17.58), one from (17.61), and one from (17.63), we have

$$-\frac{1}{2}F^\lambda_\mu \partial_\lambda A_\nu - \frac{1}{2}F^\lambda_\nu \partial_\lambda A_\mu + \frac{1}{2}F^\lambda_\mu \partial_\nu A_\lambda + \frac{1}{2}F_{\mu\tau}\partial_\nu A^\tau + \frac{1}{2}F^\tau_\mu B_{\nu\tau}. \qquad (17.65)$$

The fourth term may be rewritten as follows

$$\frac{1}{2}F_{\mu\tau}\partial_\nu A^\tau = -\frac{1}{2}F_{\tau\mu}\partial_\nu A^\tau = -\frac{1}{2}F^\lambda_\mu \partial_\nu A_\lambda$$

which just cancels the third term in (17.65). For the remaining terms, after using (17.44), we have

$$\begin{aligned}&-\frac{1}{2}F^\lambda_\mu \partial_\lambda A_\nu - \frac{1}{2}F^\lambda_\nu \partial_\lambda A_\mu + \frac{1}{2}F^\lambda_\mu \partial_\nu A_\lambda + \frac{1}{2}F^\lambda_\mu \partial_\lambda A_\nu \\ &= -\frac{1}{2}F^\lambda_\nu \partial_\lambda A_\mu + \frac{1}{2}F^\lambda_\mu \partial_\nu A_\lambda \\ &= -\frac{1}{2}F^\lambda_\nu \partial_\lambda A_\mu + \frac{1}{2}F^\lambda_\nu \partial_\mu A_\lambda - \frac{1}{2}F^\lambda_\nu \partial_\mu A_\lambda + \frac{1}{2}F^\lambda_\mu \partial_\nu A_\lambda \\ &= -\frac{1}{2}F^\lambda_\nu F_{\lambda\mu} \qquad (17.66)\end{aligned}$$

where the final result comes from the combination of the first two terms; we drop the last two terms because they are antisymmetric with respect to (μ, ν) as the Ricci tensor must be symmetric.

Collecting all terms of the type $FFAA$: the last two terms in (17.59), the last four from (17.60), the last two from (17.62), two from (17.63), and one from (17.64), we have (not displaying the common $1/4$ coefficients)

$$\begin{aligned}
&F^{\lambda}_{\tau}F^{\tau}_{\mu}A_{\lambda}A_{\nu} + F^{\lambda}_{\tau}F^{\tau}_{\nu}A_{\lambda}A_{\mu} - F^{\lambda}_{\nu}F^{\tau}_{\mu}A_{\tau}A_{\lambda} - F^{\lambda}_{\tau}F^{\tau}_{\mu}A_{\nu}A_{\lambda} \\
&-F^{\lambda}_{\tau}F^{\tau}_{\lambda}A_{\nu}A_{\mu} - F^{\lambda}_{\nu}F^{\tau}_{\lambda}A_{\tau}A_{\mu} - F_{\lambda\tau}F^{\tau}_{\mu}A^{\lambda}A_{\nu} - F_{\lambda\tau}F^{\tau}_{\nu}A^{\lambda}A_{\mu} \\
&-2F_{\nu\rho}F^{\tau}_{\mu}A^{\rho}A_{\tau} - 2F_{\tau\rho}F^{\tau}_{\mu}A^{\rho}A_{\nu} - F_{\nu\rho}F_{\mu\lambda}A^{\rho}A^{\lambda}.
\end{aligned} \tag{17.67}$$

Grouping similar terms

$$\begin{aligned}
&F^{\lambda}_{\tau}F^{\tau}_{\mu}A_{\lambda}A_{\nu} - F^{\lambda}_{\tau}F^{\tau}_{\mu}A_{\nu}A_{\lambda} - F_{\lambda\tau}F^{\tau}_{\mu}A^{\lambda}A_{\nu} - 2F_{\tau\rho}F^{\tau}_{\mu}A^{\rho}A_{\nu} \\
&= -F_{\lambda\tau}F^{\tau}_{\mu}A^{\lambda}A_{\nu} + 2F_{\lambda\tau}F^{\tau}_{\mu}A^{\lambda}A_{\nu} = F_{\lambda\tau}F^{\tau}_{\mu}A^{\lambda}A_{\nu},
\end{aligned} \tag{17.68}$$

and

$$F^{\lambda}_{\tau}F^{\tau}_{\nu}A_{\lambda}A_{\mu} - F^{\lambda}_{\nu}F^{\tau}_{\lambda}A_{\tau}A_{\mu} - F_{\lambda\tau}F^{\tau}_{\nu}A^{\lambda}A_{\mu} = -F_{\lambda\tau}F^{\tau}_{\nu}A^{\lambda}A_{\mu}, \tag{17.69}$$

again we note that the results from (17.68) and (17.69) form an antisymmetric combination in (μ, ν), hence can be dropped. After eliminating seven out of the 11 terms in (17.67) we have four left, with three of them mutually canceling:

$$\begin{aligned}
&-F^{\lambda}_{\nu}F^{\tau}_{\mu}A_{\tau}A_{\lambda} - 2F_{\nu\rho}F^{\tau}_{\mu}A^{\rho}A_{\tau} - F_{\nu\rho}F_{\mu\lambda}A^{\rho}A^{\lambda} \\
&= -F^{\lambda}_{\nu}F^{\tau}_{\mu}A_{\tau}A_{\lambda} + 2F^{\tau}_{\nu}F^{\lambda}_{\mu}A_{\tau}A_{\lambda} - F^{\tau}_{\nu}F^{\lambda}_{\mu}A_{\tau}A_{\lambda} = 0.
\end{aligned}$$

The only nonvanishing term from (17.67) is (and putting back the $1/4$ factor)

$$-\frac{1}{4}F^{\lambda}_{\tau}F^{\tau}_{\lambda}A_{\nu}A_{\mu} = \frac{1}{4}F_{\lambda\tau}F^{\lambda\tau}A_{\nu}A_{\mu}. \tag{17.70}$$

Collecting the remaining terms from (17.57)–(17.64), we have (not displaying the common $-1/2$ coefficients)

$$\begin{aligned}
&A_{\nu}\partial_{\lambda}F^{\lambda}_{\mu} + A_{\mu}\partial_{\lambda}F^{\lambda}_{\nu} \underleftrightarrow{- A_{\lambda}\partial_{\nu}F^{\lambda}_{\mu}} \underbrace{+ A_{\lambda}\Gamma^{\tau}_{\mu\nu}F^{\lambda}_{\tau}} + A_{\nu}\Gamma^{\lambda}_{\lambda\tau}F^{\tau}_{\mu} \\
&+ A_{\mu}\Gamma^{\lambda}_{\lambda\tau}F^{\tau}_{\nu} \underline{- A_{\tau}\Gamma^{\tau}_{\mu\lambda}F^{\lambda}_{\nu}} - A_{\nu}\Gamma^{\tau}_{\mu\lambda}F^{\lambda}_{\tau} \underline{- A_{\lambda}\Gamma^{\lambda}_{\nu\tau}F^{\tau}_{\mu}} \\
&- A_{\mu}\Gamma^{\lambda}_{\nu\tau}F^{\tau}_{\lambda} \underleftrightarrow{- A^{\tau}\partial_{\nu}F_{\mu\tau}} \underbrace{+ A^{\lambda}\Gamma^{\tau}_{\mu\nu}F_{\tau\lambda}} \underline{+ 2A_{\rho}\Gamma^{\rho}_{\nu\tau}F^{\tau}_{\mu}}.
\end{aligned} \tag{17.71}$$

The terms with the same underlines in (17.71) mutually cancel; we are left with six terms that can be grouped into two combinations (putting back the $-1/2$ coefficients):

$$-\frac{1}{2}\left(A_{\nu}\partial_{\lambda}F^{\lambda}_{\mu} - A_{\nu}\Gamma^{\tau}_{\mu\lambda}F^{\lambda}_{\tau} + A_{\nu}\Gamma^{\lambda}_{\lambda\tau}F^{\tau}_{\mu}\right) = -\frac{1}{2}A_{\nu}D_{\lambda}F^{\lambda}_{\mu} \tag{17.72}$$

and

$$-\frac{1}{2}\left(A_{\mu}\partial_{\lambda}F^{\lambda}_{\nu} - A_{\mu}\Gamma^{\lambda}_{\nu\tau}F^{\tau}_{\lambda} + A_{\mu}\Gamma^{\lambda}_{\lambda\tau}F^{\tau}_{\nu}\right) = -\frac{1}{2}A_{\mu}D_{\lambda}F^{\lambda}_{\nu} \tag{17.73}$$

where D_{λ} is the covariant derivative.[14]

[14]Recall from Eq. (13.43) that the covariant derivative of a tensor with multiple indices (i.e. tensor of higher rank) has a Christoffel factor for each index.

Collecting the results from (17.66), (17.70), (17.72), and (17.73) we have, after putting back the $R_{\mu\nu}$ discussed just before our calculation of the factors (1)–(8),

$$\hat{R}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}F^{\lambda}_{\nu}F_{\lambda\mu} + \frac{1}{4}F_{\lambda\tau}F^{\lambda\tau}A_{\nu}A_{\mu} - \frac{1}{2}\left(A_{\nu}D_{\lambda}F^{\lambda}_{\mu} + A_{\mu}D_{\lambda}F^{\lambda}_{\nu}\right). \quad (17.74)$$

The 5D Ricci tensor components $\hat{R}_{5\nu}$ and $\hat{R}_{55}$

Equation (17.18) with $M = 5$ and $N = \nu$ can now be written as

$$\hat{R}_{5\nu} = \partial_L \hat{\Gamma}^L_{5\nu} - \partial_\nu \hat{\Gamma}^L_{5L} + \hat{\Gamma}^L_{LT}\hat{\Gamma}^T_{5\nu} - \hat{\Gamma}^L_{\nu T}\hat{\Gamma}^T_{5L}. \quad (17.75)$$

There are two pairs of repeated indices, L and T. We will now consider the separate cases when they take on 4D values of $L = \lambda$ and $T = \tau$, or the extra dimensional index value of 5.

1. $L = \lambda$ and $T = \tau$:

$$\left(\hat{R}_{5\nu}\right)_1 = \partial_\lambda \hat{\Gamma}^\lambda_{5\nu} - \underline{\partial_\nu \hat{\Gamma}^\lambda_{5\lambda}} + \hat{\Gamma}^\lambda_{\lambda\tau}\hat{\Gamma}^\tau_{5\nu} - \hat{\Gamma}^\lambda_{\nu\tau}\hat{\Gamma}^\tau_{5\lambda}. \quad (17.76)$$

2. $L = 5$ and $T = \tau$:

$$\left(\hat{R}_{5\nu}\right)_2 = \underline{\partial_5 \hat{\Gamma}^5_{5\nu}} - \underline{\partial_\nu \hat{\Gamma}^5_{55}} + \underbrace{\hat{\Gamma}^5_{5\tau}\hat{\Gamma}^\tau_{5\nu}} - \underline{\hat{\Gamma}^5_{\nu\tau}\hat{\Gamma}^\tau_{55}}. \quad (17.77)$$

3. $L = \lambda$ and $T = 5$:

$$\left(\hat{R}_{5\nu}\right)_3 = \underline{\hat{\Gamma}^\lambda_{\lambda 5}\hat{\Gamma}^5_{5\nu}} - \underbrace{\hat{\Gamma}^\lambda_{\nu 5}\hat{\Gamma}^5_{5\lambda}}. \quad (17.78)$$

4. $L = T = 5$:

$$\left(\hat{R}_{5\nu}\right)_4 = \underline{\hat{\Gamma}^5_{55}\hat{\Gamma}^5_{5\nu}} - \underline{\hat{\Gamma}^5_{\nu 5}\hat{\Gamma}^5_{55}}. \quad (17.79)$$

Those with straight underlines will individually vanish by themselves, while the two with braces under them cancel each other. We are left with

$$\hat{R}_{5\nu} = \partial_\lambda \hat{\Gamma}^\lambda_{5\nu} + \hat{\Gamma}^\lambda_{\lambda\tau}\hat{\Gamma}^\tau_{5\nu} - \hat{\Gamma}^\lambda_{\nu\tau}\hat{\Gamma}^\tau_{5\lambda}. \quad (17.80)$$

According to (17.52), $\hat{\Gamma}^\lambda_{5\nu} = -\frac{1}{2}F^\lambda_\nu$ and $\hat{\Gamma}^\lambda_{\nu\tau} = \Gamma^\lambda_{\nu\tau} - \frac{1}{2}\left(F^\lambda_\nu A_\tau + F^\lambda_\tau A_\nu\right)$ so that $\hat{\Gamma}^\lambda_{\nu\lambda} = \Gamma^\lambda_{\nu\lambda} - \frac{1}{2}\left(F^\lambda_\nu A_\lambda + F^\lambda_\lambda A_\nu\right) = \Gamma^\lambda_{\nu\lambda} - \frac{1}{2}F^\lambda_\nu A_\lambda$, and we have, when adding up all the nonvanishing terms from (17.76) to (17.79),

$$\begin{aligned} \hat{R}_{5\nu} &= -\frac{1}{2}\left(\partial_\lambda F^\lambda_\nu + \Gamma^\lambda_{\lambda\tau}F^\tau_\nu - \Gamma^\lambda_{\nu\tau}F^\tau_\lambda\right) \\ &\quad + \frac{1}{4}A_\lambda F^\lambda_\tau F^\tau_\nu - \frac{1}{4}\left(F^\lambda_\nu A_\tau + F^\lambda_\tau A_\nu\right)F^\tau_\lambda \\ &= -\frac{1}{2}D_\lambda F^\lambda_\nu - \frac{1}{4}A_\nu F^\lambda_\tau F^\tau_\lambda \end{aligned} \quad (17.81)$$

because two middle terms cancel each other.

Finally, we need to calculate $\hat{R}_{55}$. Just setting $\nu = 5$ in (17.80) we have

$$\hat{R}_{55} = \partial_\lambda \hat{\Gamma}^\lambda_{55} + \hat{\Gamma}^\lambda_{\lambda\tau}\hat{\Gamma}^\tau_{55} - \hat{\Gamma}^\lambda_{5\tau}\hat{\Gamma}^\tau_{5\lambda}. \tag{17.82}$$

Since $\hat{\Gamma}^\lambda_{55} = 0$, we obtain the simple expression

$$\hat{R}_{55} = -\frac{1}{4}F^\lambda_\tau F^\tau_\lambda = +\frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{17.83}$$

17.5.3 From 5D Ricci tensor to 5D Ricci scalar

Separating out the 5D spacetime index $M = (\mu, 5)$ into the 4D plus the extra dimensional indices, the Ricci scalar of (17.19) is seen to be composed of three terms:

$$\hat{R} = \hat{g}^{MN}\hat{R}_{MN} = \hat{g}^{\mu\nu}\hat{R}_{\mu\nu} + 2\hat{g}^{5\nu}\hat{R}_{5\nu} + \hat{g}^{55}\hat{R}_{55}. \tag{17.84}$$

Collecting all the $\hat{R}_{MN}$ components in one place, we have from (17.74), (17.81), and (17.83):

$$\begin{aligned}
\hat{R}_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}F^\lambda_\nu F_{\lambda\mu} + \frac{1}{4}F_{\lambda\tau}F^{\lambda\tau}A_\nu A_\mu - \frac{1}{2}\left(A_\nu D_\lambda F^\lambda_\mu + A_\mu D_\lambda F^\lambda_\nu\right)\\
\hat{R}_{5\nu} &= -\frac{1}{2}D_\lambda F^\lambda_\nu - \frac{1}{4}A_\nu F^\lambda_\tau F^\tau_\lambda \\
\hat{R}_{55} &= +\frac{1}{4}F_{\mu\nu}F^{\mu\nu}.
\end{aligned} \tag{17.85}$$

From Eqs. (17.84) and (17.85) and the inverse metric elements of (17.14), we have for the 5D Ricci scalar

$$\begin{aligned}
\hat{R} &= \hat{g}^{\mu\nu}\hat{R}_{\mu\nu} + 2\hat{g}^{5\nu}\hat{R}_{5\nu} + \hat{g}^{55}\hat{R}_{55}\\
&= g^{\mu\nu}\left[R_{\mu\nu} - \frac{1}{2}F^\lambda_\nu F_{\lambda\mu} + \frac{1}{4}F_{\lambda\tau}F^{\lambda\tau}A_\nu A_\mu - \frac{1}{2}\left(A_\nu D_\lambda F^\lambda_\mu + A_\mu D_\lambda F^\lambda_\nu\right)\right]\\
&\quad + A^\nu D_\lambda F^\lambda_\nu + \frac{1}{2}A^\nu A_\nu F^\lambda_\tau F^\tau_\lambda + \frac{1}{4}\left(1 + A^\lambda A_\lambda\right)F_{\mu\nu}F^{\mu\nu}.
\end{aligned} \tag{17.86}$$

The F^2A^2 terms cancel

$$\frac{1}{4}F_{\lambda\tau}F^{\lambda\tau}A_\nu A^\nu + \frac{1}{2}A^\nu A_\nu F_{\lambda\tau}F^{\tau\lambda} + \frac{1}{4}A^\lambda A_\lambda F_{\mu\nu}F^{\mu\nu} = 0. \tag{17.87}$$

The ADF terms cancel because

$$\begin{aligned}
&-\frac{1}{2}g^{\mu\nu}\left(A_\nu D_\lambda F^\lambda_\mu + A_\mu D_\lambda F^\lambda_\nu\right) + A^\nu D_\lambda F^\lambda_\nu\\
&= -\frac{1}{2}A^\mu D_\lambda F^\lambda_\mu - \frac{1}{2}A^\nu D_\lambda F^\lambda_\nu + A^\nu D_\lambda F^\lambda_\nu = 0.
\end{aligned} \tag{17.88}$$

Remarkably, with the Kaluza postulate for the 5D metric $\hat{g}_{MN}$ of (17.11), the resultant 5D Ricci scalar (17.86) reduces to the simple expression of (17.20):

$$\hat{R} = R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{17.89}$$

Some people call this the **Kaluza–Klein miracle**.