Waves, particles, and quantum jumps

6

- The idea of light quanta was developed further by Einstein in 1909. Through a study of radiation energy fluctuation, he proposed that light had the complementary property of wave and particle. This is the first statement ever on wave–particle duality.
- In this discussion, Einstein suggested that the energy quanta were carried by point-like particles—what he termed then as "the point of view of Newtonian emission theory"—what we now call the photon.
- We describe the parallel development in spectroscopy that eventually led to Bohr's quantum model of atomic structure. He postulated that, like radiation, atoms also have quantized energies with transitions characterized by quantum jumps; this led him to the successful explanation of the hydrogen spectrum.
- In three overlapping but nonidentical papers in 1916–17, Einstein used Bohr's quantum jump idea to construct a microscopic theory of radiation–matter interaction. Through what came to be known as Einstein's *A* and *B* coefficients, he showed how Planck's spectral distribution followed. The central novelty and lasting feature is the introduction of probability in quantum dynamics.
- In Section 6.4, we present a brief introduction to quantum field theory. The treatment of the harmonic oscillator in the new quantum mechanics is reviewed. A quantized field is a collection of quantum oscillators. We show that the Planck/Einstein quantization result is automatically obtained in this new theoretical framework. This had at last put Einstein's idea of the photon on a firm mathematical foundation.
- The noncommutivity of physical observables in the new quantum theory brings about features that can be identified as creation and annihilation of quantum states. This gives a natural description of the quantum jumps of radiation emission and absorption. In fact they can be extended to the depiction of creation and destruction of material particles as well—a key characteristic of interactions at relativistic energies.
- Finally we explain how the wave–particle duality first discovered by Einstein in the study of radiation energy fluctuation is resolved in quantum field theory.

As we have already mentioned in Chapter 4, it would still be some years before Einstein openly committed himself to a point-like particle interpretation of light quanta. In Section 6.1 we shall discuss Einstein's 1909 study of radiation fluctuation that led him to show, for the first time, that light had not just wave or just particle properties, but a sort of fusion of the two—what came to be known as "wave–particle duality". The next big event in quantum history was the 1913 model for the structure of the atom conceived by Niels Bohr, who applied the Planck/Einstein quantum to the study of the hydrogen spectrum (Bohr 1913). Its spectacular success in effect launched a new era in exploration of the quantum world—what we now call the 'old quantum theory'. Bohr's 'quantum jumps' $E_i - E_f = h\nu$ inspired Einstein in 1916 to propose a detailed study of the radiation mechanism that takes place in a blackbody radiation cavity. He introduced his famous A and B coefficients for the theory of stimulated and spontaneous emissions of radiation. This is the first time that a probability description was invoked in the description of quantum dynamics, and it presaged some of the surprising consequences that would be obtained later in quantum mechanics and quantum field theory. In Section 6.4 we shall present some of the basic elements of quantum field theory to see how it is capable of resolving in one elegant framework the apparent contradictions of waves, particles, and quantum jumps. But Einstein never accepted this new paradigm.

6.1 Wave–particle duality

We have pointed out that Planck did not himself consider the quantum of action as relating directly to any physical entity, and the light quantum proposal of Einstein met considerable resistance from the physics community. This resistance can best be illustrated by the attitude of Robert Millikan, who spent a decade verifying Einstein's prediction for the photoelectric effect. Describing his viewpoint in later years, Millikan wrote this way: "I spent ten years of my life testing that 1905 equation of Einstein's, and contrary to all my expectations, I was compelled in 1915 to assert its unambiguous experimental verification in spite of its unreasonableness since it seemed to violate everything that we knew about the interference of light" (Millikan 1949).

When the idea of the light quantum $\epsilon = h\nu$ was proposed in 1905, there was still the question as what forms the quantum would take. There is the possibility that the energy is distributed throughout space as is the case with waves, or as discontinuous lumps of energy, like particles. By 1909 Einstein was more explicit in proposing that light in certain circumstances was composed of particles (Einstein 1909a,b)—in contradiction to the well-established wave properties of light. Waves cannot have particle properties and particles cannot behave like waves. However, even without the detailed knowledge of quantum electrodynamics, Einstein was able to make some definite statements on the nature of light (wave vs. particle). His argument was based on a study of the energy fluctuations in radiation. Einstein showed that light was neither simply waves nor simply particles, but had the property of being both waves and particles at the same time. The notion of wave–particle duality was born.

6.1.1 Fluctuation theory (Einstein 1904)

In Chapter 2 we discussed Einstein's theory of Brownian motion, which involved the investigation of the fluctuation phenomenon. This is a subject that had long interested Einstein. According to Boltzmann's distribution, the average energy is given by

$$\langle E \rangle = \frac{\int E e^{-\gamma E} \omega(E)\, dE}{\int e^{-\gamma E} \omega(E)\, dE} \tag{6.1}$$

with $\omega(E)$ being the density of states having energy E and $\gamma = (k_B T)^{-1}$. Einstein in 1904 found the following fluctuation relation, after making the differentiation $-\partial/\partial\gamma$ of $\langle E \rangle$ in (6.1):

$$\left\langle \Delta E^2 \right\rangle = k_B T^2 \frac{\partial \langle E \rangle}{\partial T}, \tag{6.2}$$

where

$$\left\langle \Delta E^2 \right\rangle \equiv \left\langle (E - \langle E \rangle)^2 \right\rangle = \left\langle E^2 \right\rangle - \langle E \rangle^2 \tag{6.3}$$

is the square deviation from the mean (the variance).

In 1904 Einstein was interested in finding systems with large fluctuations: $\left\langle \Delta E^2 \right\rangle \simeq \langle E \rangle^2$, and he studied the volume dependence of such a system. It is plausible to conclude that such an investigation led him to delve into the volume dependence of radiation entropy, which (as we have shown in Chapter 4) was the crucial step in his arriving in 1905 at the idea of light quanta. A study of the fluctuation theory is also instrumental in his finally arriving at the view that light quanta are point-like particles.

6.1.2 Energy fluctuation of radiation (Einstein 1909a)

Consider a small volume $\tilde{v}$, immersed in thermal radiation (see Fig. 6.1) having energy in the frequency interval $(\nu, \nu + d\nu)$ as

$$\langle E \rangle = \tilde{v} \rho(\nu, T)\, d\nu \tag{6.4}$$

(cf. the original definition of radiation energy density ρ given in Section 3.2.3). In his 1909 papers, Einstein used (6.2) to calculate the variance from the various radiation density distributions $\rho(\nu, T)$. For this small volume one obtains

$$\left\langle \Delta E^2 \right\rangle = \tilde{v} k_B T^2 d\nu \frac{\partial \rho}{\partial T}. \tag{6.5}$$

This general result holds whether the system is randomly distributed as waves or particles, because it is based on Boltzmann's principle and on the fact that the spectral density at a given frequency depends on temperature only. The fluctuation formulas for the different distribution laws are presented below.

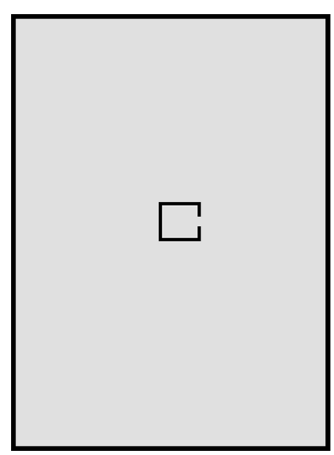

Fig. 6.1 A small volume $\tilde{v}$ immersed in thermal radiation at temperature T.

Radiation in the Rayleigh–Jeans limit fluctuates like waves

For the radiation described by the Rayleigh–Jeans distribution (4.3)

$$\rho_{\rm RJ} = \frac{8\pi\nu^2}{c^3} k_{\rm B} T, \tag{6.6}$$

the fluctuation formula (6.5) leads to

$$\left\langle \Delta E^2 \right\rangle_{\rm RJ} = \tilde{v} \frac{8\pi\nu^2}{c^3} k_{\rm B}^2 T^2 d\nu = \frac{c^3}{8\pi\nu^2} \frac{\langle E \rangle^2}{\tilde{v} d\nu}. \tag{6.7}$$

To reach the last expression we have used (6.6) and (6.4) to replace temperature by the average energy. In the following we shall argue that such a variance reflects the wave nature of the system.

Fluctuations of a wave system A system of randomly mixed waves should display fluctuations. Although the light in an enclosure is distributed uniformly, at a certain point in space and time a light wave of a certain frequency may interfere, constructively or destructively, with another wave of slightly different frequency. This beat phenomenon would cause the energy in this small volume to be larger or smaller than the average value. The result given in (6.7) just reflects a fluctuating wave system. The key feature of wave fluctuation is that, for each radiation oscillator (i.e. degree of freedom, or mode), we have the remarkable result (derived in SuppMat Section 6.5) that the fluctuation in the energy density $\sqrt{\Delta u^2}$ has the same magnitude as the (average) energy density u itself:

$$\sqrt{\Delta u^2} = u. \tag{6.8}$$

This result can be translated into the variance and average energy of the wave system by a consideration of the involved degrees of freedom. The average energy of the system $\langle E \rangle$ requires the summation of all modes, thus a multiplication of the oscillator number in the $(\nu, \nu + d\nu)$ interval $N d\nu$ and the average energy density u for each oscillator:

$$\langle E \rangle = N d\nu u. \tag{6.9}$$

The calculation of the variance $\left\langle \Delta E^2 \right\rangle$ involves a similar sum, i.e. the same multiplication factor, $\left\langle \Delta E^2 \right\rangle = N d\nu \Delta u^2$. The result in (6.8) then implies

$$\left\langle \Delta E^2 \right\rangle = \frac{\langle E \rangle^2}{N d\nu}. \tag{6.10}$$

We have already calculated the wave mode number in Chapter 4 as displayed in (4.5):

$$N d\nu = \frac{8\pi\nu^2}{c^3} \tilde{v} d\nu. \tag{6.11}$$

Substituting this expression into (6.10) we obtain a result in agreement with the relation (6.7). This wave fluctuation result is to be expected as the Rayleigh–Jeans law follows from the classical Maxwell wave theory.

Radiation in the Wien limit fluctuates like particles

For radiation described by the Wien distribution, cf. Eq. (4.12),

$$\rho_{\mathrm{W}} = \frac{8\pi h\nu^3}{c^3} e^{-h\nu/k_{\mathrm{B}}T} \tag{6.12}$$

we have $\langle E \rangle = \tilde{v}\rho_{\mathrm{W}} d\nu$ and, from formula (6.5), the fluctuation result

$$\left\langle \Delta E^2 \right\rangle_{\mathrm{W}} = \tilde{v} h\nu \frac{8\pi h\nu^3}{c^3} e^{-h\nu/k_{\mathrm{B}}T} d\nu = h\nu \langle E \rangle, \tag{6.13}$$

which is clearly different from result expected from fluctuation of system of waves. In fact the fractional fluctuation has the form

$$\frac{\sqrt{\left\langle \Delta E^2 \right\rangle}}{\langle E \rangle} = \sqrt{\frac{h\nu}{\langle E \rangle}}. \tag{6.14}$$

This is exactly the fluctuation that one would expect of a system of particles. We have already discussed such a situation in Chapter 2 on Brownian motion. In particular we have shown that Brownian motion can be modeled as random walks. Equation (2.14) demonstrates that any system of random discrete entities would have a fractional deviation of $N^{-1/2}$ as is the case displayed in (6.14) because, in our case, we have $\langle E \rangle = Nh\nu$.

This result then strengthened Einstein's original proposal that blackbody radiation in the Wien limit behaves statistically like a gas of photons.

Planck distribution: Radiation fluctuates like particles *and* waves

Observationally, radiation is correctly described throughout its frequency range by the Planck spectral law. We now calculate the energy fluctuation from Planck's distribution

$$\rho = \frac{8\pi h}{c^3} \frac{\nu^3}{\exp(h\nu/k_{\mathrm{B}}T) - 1}. \tag{6.15}$$

Remarkably we find the result is simply the sum of two terms, one being the Rayleigh–Jeans terms of (6.7) and the other being the Wien term of (6.13):

$$\left\langle \Delta E^2 \right\rangle_{\mathrm{P}} = \left\langle \Delta E^2 \right\rangle_{\mathrm{RJ}} + \left\langle \Delta E^2 \right\rangle_{\mathrm{W}}. \tag{6.16}$$

This shows that radiation is neither simply waves nor simply particles. This led Einstein to suggest in 1909 that radiation can be viewed as a "fusion" of waves and particles.

Einstein proceeded to the calculation of the pressure fluctuation using explicitly the particle property of a light quantum: a photon has momentum $p = h\nu/c$. Thus, together with the suggestion of light's dual nature, Einstein now stated for the first time his view that quanta were carried by point-like particles.

6.2 Bohr's atom—another great triumph of the quantum postulate

While quantum theory has its origin in the study of blackbody radiation, there was also a parallel development in spectroscopy of the radiation emitted and absorbed by atoms. Bohr's quantum model of the hydrogen atom brought great success in this area. Thus, together with blackbody radiation, they formed the twin foundations of the quantum theory.[1]

[1]For a clear exposition of the 'old quantum theory', we recommend Tomonaga (1962).

6.2.1 Spectroscopy: Balmer and Rydberg

We mentioned in Chapter 3 that, besides blackbody radiation, Gustav Kirchhoff also made major contributions in spectroscopy. But we will start our story with the Swiss high-school mathematics teacher Johann Balmer (1825 – 98). The hydrogen spectrum is particularly simple: it has four lines in the visible range: $H_\alpha = 6563\,\text{Å}$, $H_\beta = 4861\,\text{Å}$, $H_\gamma = 4341\,\text{Å}$, $H_\delta = 4102\,\text{Å}$ (Fig. 6.2). In 1885 Balmer made the remarkable discovery that these wavelengths follow a pattern when written in units of $H = 3645.6\,\text{Å}$:

$$H_\alpha = \frac{9}{5}H, \quad H_\beta = \frac{16}{12}H, \quad H_\gamma = \frac{25}{21}H, \quad \text{and} \quad H_\delta = \frac{36}{32}H.$$

He then extended this to the relation (the Balmer formula) as

$$\lambda = \frac{n^2}{n^2 - 4}H, \tag{6.17}$$

which covers the original four lines with $n = 3, 4, 5, 6$, and, as it turned out, could also account for the other lines in the ultraviolet region.

This pattern was later generalized to other hydrogen lines by Johannes Rydberg (1854–1919) in the form of

$$\frac{1}{\lambda} = R\left(\frac{1}{m^2} - \frac{1}{n^2}\right) \tag{6.18}$$

with the **Rydberg constant** $R = 4/H$ and both (m, n) being integers. The case $m = 2$ reduces to the Balmer series (visible), $m = 1$ to the Lyman series (infrared, found in 1906), and $m = 3$ to the Paschen series (ultraviolet, found in 1908).

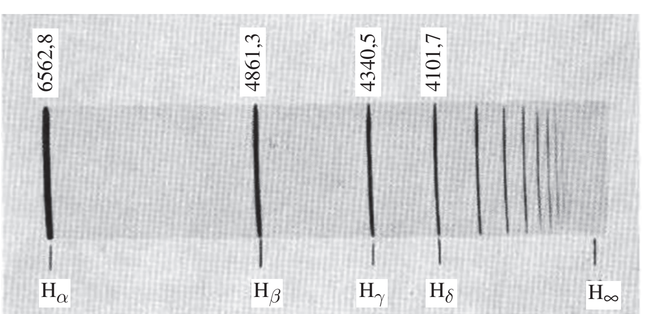

Fig. 6.2 Hydrogen spectral lines $H_\alpha, H_\beta, H_\gamma, H_\delta, \ldots$ Picture from Tomonago (1962).

6.2.2 Atomic structure: Thomson and Rutherford

The discovery of the first subatomic particle, the electron, is traditionally attributed to J.J. Thomson (1856–1940), for his measurement of the charge-to-mass ratio of cathode ray particles in 1897. He proposed a theory of atomic structure that pictures electrons as being embedded in a sphere of uniformly distributed positive charges—a sort of raisins-and-pudding model. The size of the atom had to be put in by hand as there was no way to construct any length-scale from the fundamental constants of charge and mass (e, m) from classical physics. The spectral lines are supposed to result from periodic oscillations of the electrons. However if one identifies the emission lines with the fundamental frequencies, there is no way to get rid of the unwanted higher harmonics.

During the period around 1910, Ernest Rutherford (1871–1937) and his collaborators performed a series of alpha particle scattering experiments. The large scattering angle result led Rutherford to suggest that an atom is mostly empty space, with all the positive charges concentrated in a compact center and electrons circulating around this atomic nucleus. Such a model of the atom still had the deficiencies of no natural path to an atomic size and the presence of higher harmonics. Furthermore, the circulating electrons, according to classic electromagnetism, must necessarily radiate away their energies and spiral into the nucleus. It did not seem to have a way to explain the atom's stability.

6.2.3 Bohr's quantum model and the hydrogen spectrum

Niels Bohr was familiar with Rutherford's atom. In 1913 he found a way to construct an atomic model that overcame the difficulties that Rutherford (and Thomson) had encountered. Moreover, he was able to predict in a simple way the spectrum of the hydrogen atom, with the Rydberg constant expressed in terms of fundamental constants (Bohr 1913). The new input that Bohr had was the quantum of Planck and Einstein.

Planck's constant naturally leads to an atomic scale

We have already mentioned that there is no way to construct an atomic length-scale from the two relevant constants (m, e) of classical mechanics and electromagnetism. With the introduction of Planck's constant, this can be done:

$$l = \frac{h^2}{me^2}. \tag{6.19}$$

One can easily check that this has the dimension of a length.[2] Putting in the values of the electron mass, the charge and Planck's constant (m, e, h), one finds an l of about 20 Å, roughly in the range of the atomic-scale. The Rydberg constant of (6.18) must have the dimension of inverse length, and as we shall see, it is indeed inversely proportional to the length-scale displayed here.

[2]Keeping in mind the Coulomb energy, we see that e^2 has the dimension of (energy·length). The mass m has (momentum^2/energy). Thus the denominator me^2 has (momentum^2·length). With the numerator h^2 being (momentum·length)^2, the ratio $h^2/(me^2)$ has the dimension of a length.

Stationary states and quantum jumps

Bohr reasoned that, since radiation energy is quantized, the atomic energies should similarly form a discrete set. He hypothesized that atoms should be

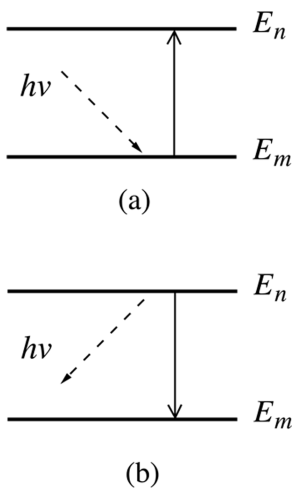

Fig. 6.3 Transitions between atomic states $n \leftrightarrow m$. (a) Absorption of a photon with energy $h\nu$. (b) Emission of a photon.

stable at these quantized values E_n, with $n = 0, 1, 2, 3, \ldots$ Namely, he postulated the existence of a set of stationary states. The absorption and emission of radiation then corresponds to 'jumps' among these quantized states. A state with energy E_m can absorb a photon of frequency ν and makes the transition to a higher energy state E_n, provided energy conservation is respected (the **Bohr frequency rule**):

$$\nu = \frac{E_n - E_m}{h}. \tag{6.20}$$

Such a transition is depicted in Fig. 6.3(a). Significantly, Bohr proposed the revolutionary concept that one must reject any attempt to visualize or to explain the behavior of the electron during the transition of the atom from one stationary state to another. In fact we can interpret Einstein's photoelectric effect as such a transition, if the kinetic energy of the final state electron is ignored. If we accept this possibility, it is then entirely natural to stipulate the inverse process: when an atom makes a downward transition $n \to m$, it should be accompanied by the emission of a photon, as pictured in Fig. 6.3(b). When the frequency rule (6.20) is expressed in terms of wavelength, we have

$$\frac{1}{\lambda} = \frac{\nu}{c} = \frac{E_n}{hc} - \frac{E_m}{hc}. \tag{6.21}$$

Comparing this with the Rydberg formula (6.18), Bohr had a way to connect the atomic energy levels to the Rydberg constant:

$$E_n = -R\frac{hc}{n^2}, \tag{6.22}$$

consistent with the initial assumption that atomic energies are quantized. The energy is negative because it is the binding energy. It is interesting to relate that Bohr was unaware of the Balmer/Rydberg formulas when he started out in his search for an atomic theory. When he was finally told of the Balmer series in 1913, it was a great revelation to him. He later recalled:[3] "As soon as I saw Balmer's formula, the whole thing was clear to me."

[3] As recounted by Heilbron (1977).

Quantization of angular momentum

Bohr then hypothesized that once in these stationary states, the electron's motion was correctly described by classical mechanics. The total energy is the sum of the kinetic and potential energies:

$$E_n = \frac{mv_n^2}{2} - \frac{e^2}{r_n}. \tag{6.23}$$

If for simplicity we take the orbits to be circles, the velocity v_n is related to the centrifugal acceleration v_n^2/r_n, which is fixed by the balance of centrifugal and Coulomb forces $mv_n^2/r_n = e^2/r_n^2$. In this way we find from (6.23) that the total energy is just one-half of the potential energy,

$$E_n = -\frac{e^2}{2r_n}. \tag{6.24}$$

This simple relation makes it clear that quantized energies imply a set of quantized orbits. The higher the energy (i.e. less negative) the larger would be the orbital radius r_n.

What determines the choice of these quantized orbits? Bohr suggested two ways to proceed and he demonstrated that both approaches led to the same conclusion. One can either make the assumption that the classical description will be valid for the description of states with large quantum number n, hence large orbits. (This became known later on as **the correspondence principle**.) Or, the same result was obtained by Bohr with the postulate of angular momentum quantization

$$L_n = n\hbar \equiv n\frac{h}{2\pi}. \tag{6.25}$$

Let us see how Bohr used angular momentum quantization[4] to deduce the quantized orbits and quantized atomic energies. The total energy can be expressed in terms of the orbital angular momentum $E = L^2/2I$. For the presently assumed circular orbits, we have (6.24) with a moment of inertia $I = mr_n^2$:

$$\frac{e^2}{2r_n} = |E_n| = \frac{L_n^2}{2mr_n^2} = n^2\frac{\hbar^2}{2mr_n^2}; \tag{6.26}$$

the last equality follows from (6.25). This fixes the radii of the quantized orbits:

$$r_n = n^2\frac{\hbar^2}{me^2} = n^2 a \tag{6.27}$$

where $a = \hbar^2/(me^2)$ is the **Bohr radius**—just the atomic-scale l of (6.19) divided by $(2\pi)^2$. We can use (6.24) to translate this into the atomic energy

$$E_n = -\frac{e^2}{2a}\frac{1}{n^2}. \tag{6.28}$$

This in turn predicts, through (6.22), the Rydberg constant to be

$$R = -\frac{n^2}{hc}E_n = \frac{e^2/\hbar c}{4\pi}\frac{1}{a} = \frac{\alpha}{4\pi}\frac{1}{a}, \tag{6.29}$$

where we have introduced the shorthand, **fine structure constant** $\alpha = e^2/\hbar c \simeq 1/137$. Putting back all the fundamental constants of (m, e, h), we have

$$R = \frac{2\pi^2 me^4}{ch^3}, \tag{6.30}$$

which was in good agreement with the experimental value of R.

One more bit of interesting history—a sort of icing-on-the-cake (cf. Section 15.6, Longair 2003). One of the first applications made by Bohr of his new theory was to explain the lines in the observed spectrum of the star ς-Puppis. They were thought to be hydrogen lines because of their similarity to the Balmer series. Bohr showed they were really those of the singly ionized helium He^+ which according to the new theory should have exactly the same spectrum as

[4] In Bohr's 1913 paper, he acknowledged that J.W. Nicholson was the first one to discover in 1912 the quantization of angular momentum. We also note that, when the circular electron orbit assumption is relaxed to allow for elliptical trajectories, as first done by Arnold Sommerfeld, the quantum numbers must be extended, besides the principal quantum number n, to include the orbital quantum number $l = 0, 1, \ldots, n-1$.

hydrogen with only the Rydberg constant being four times larger $R_{He}/R_H = 4$—as the factor e^4 in (6.30) has to be replaced by $(Ze^2)^2$ with the atomic number $Z = 2$ for helium. But it was pointed out to him that the experimental value was not exactly 4 but 4.00163. Bohr then realized that the electron mass m in (6.30) should more accurately be the 'reduced mass' $\mu = m_e m_N/(m_e + m_N)$ when the finite nuclear mass m_N was taken into account. Since the helium nucleus is four times larger than the hydrogen nucleus, one then has a ratio of Rydberg constants that is in much closer agreement with observation:

$$\frac{R_{He}}{R_H} = 4\frac{1 + m_e/m_N}{1 + m_e/4m_N} = 4.00160. \tag{6.31}$$

Here is another instance of the importance of high-precision measurements!

6.3 Einstein's *A* and *B* coefficients

During the five-year period prior to 1916, Einstein was preoccupied with the development of general relativity (see Chapters 12–14), which he finalized in 1915–16. In late 1916 he returned his attention to the study of quantum theory (Einstein 1916b,c). Having been inspired by Bohr's papers,[5] he obtained new insights into the microscopic physics concerning the emission and absorption of radiation. In constructing his theory of atomic structure Bohr had used Einstein's quantum idea, which was originally obtained from a thermal statistical study of blackbody radiation. Now Einstein used Bohr's idea of quantum jumps (Fig. 6.3) to construct a microscopic theory of the emission and absorption of radiation by molecular states to show that the resulting radiation distribution is just the Planck spectral law. He found that he could obtain Planck's spectral distribution if, and only if, the quantum jump between two molecular states $m \rightleftarrows n$ involved a monochromatic energy quantum obeying Bohr's frequency condition (6.20). Notably, Einstein's 1916 theory involved the introduction, for the first time, of a probabilistic description of quantum dynamics.

[5]When Einstein heard about Bohr's result on astrophysical helium spectrum in a meeting in Vienna in September 1913, he was astonished and said: 'Then the frequency of light does not depend at all on the orbiting frequency of the electron. And this is an *enormous achievement*. The theory of Bohr must be right.' (see p. 137, Moore 1989).

Furthermore, Einstein showed that, if the radiation is pictured as a collection of particles, the energy exchange $\Delta\epsilon = h\nu$ between molecules and radiation would also entail the exchange of momentum. For massless photons, relativity dictates a momentum transfer of $\Delta p = h\nu/c$. In this way he showed that the Planck distribution of radiation energy is precisely compatible with a Maxwell velocity distribution for the molecules. The results Einstein obtained in this investigation, in particular those related to stimulated emission of radiation, laid the foundation for the later invention of the laser and maser. Another aspect of the work was the forerunner of the theory of quantum vacuum fluctuation.

6.3.1 Probability introduced in quantum dynamics

Einstein considered a system in thermal equilibrium, consisting of a gas of particles (called molecules) and electromagnetic radiation (with spectral density ρ). Let $\epsilon_1, \epsilon_2, \epsilon_3, \ldots$ be the energies of the molecular states. The relative probability of molecules in the different states is given by Boltzmann's statistics as

$$P_n = g_n e^{-\epsilon_n/k_B T} \tag{6.32}$$

where g_n is the number of states having the same energy ϵ_n (called the degeneracy of the state). The various interactions between radiation with these molecules are considered. The two molecular states with energies $\epsilon_n > \epsilon_m$, as depicted in Fig. 6.3, will be the focus of the following discussion.

Spontaneous emission

Consider first the emission of a photon with the molecule making the $n \to m$ transition as depicted in Fig. 6.3(b). Here Einstein introduced a probabilistic description. He argued that since it is possible for a classical oscillator to radiate without the excitation (i.e. without any perturbation) by an external radiation field, the rate of the probability change (the change of the molecular number) for this spontaneous emission may be written as

$$\left(\frac{dP_n}{dt}\right)_{\text{sp-em}} = A_n^m P_n \tag{6.33}$$

where A_n^m is a constant with the lower index denoting the initial state, and the upper index the final state. Einstein noted that this mechanism of spontaneous emission of radiation is generally identical to Rutherford's 1900 statistical description of spontaneous decay of radiative matter. While Einstein could not explain the puzzle of a statistical theory he was the first one to note that it could only be understood in the quantum-theoretical context. Furthermore, Einstein immediately expressed his misgiving that such a probabilistic description seemed to imply an abandonment of strict causality.

Stimulated absorption and emission

In a field of radiation, a molecular oscillator changes its energy because the radiation transfers energy to the oscillator. Depending on the phases of the molecular oscillator and the oscillating electromagnetic field, the transferred work can be positive (absorption) or negative (emission). We call such a processes 'induced' or 'stimulated' because of the presence of the radiation perturbation. We expect the rate of change to be proportional to the radiation density ρ. For the induced absorption, we denote the molecular number change by

$$\left(\frac{dP_m}{dt}\right)_{\text{abs}} = B_m^n \rho P_m. \tag{6.34}$$

Similarly for the stimulated emission, we have

$$\left(\frac{dP_n}{dt}\right)_{\text{st-em}} = B_n^m \rho P_n. \tag{6.35}$$

The radiation density is fixed to be Planck's distribution

What is the form of the radiation spectral density such that it is compatible with this microscopic description of radiation–matter interaction? To reach equilibrium, the absorption and emission rates must balance out:

$$\left(\frac{dP}{dt}\right)_{\text{abs}} = \left(\frac{dP}{dt}\right)_{\text{st-em}} + \left(\frac{dP}{dt}\right)_{\text{sp-em}} \tag{6.36}$$

or

$$g_m B_m^n e^{-\epsilon_m/k_B T} \rho = g_n e^{-\epsilon_n/k_B T} \left(B_n^m \rho + A_n^m\right). \tag{6.37}$$

We further assume that the energy density ρ goes to infinity as the temperature increases to infinity ($T \to \infty$). The large ρ factor means that we can ignore the A_n^m term in the parentheses; in this way we obtain[6]

[6] In Einstein's original paper this was justified by the experimental condition that for large temperature ($\nu/T \to 0$) the spectral density $\rho \sim \nu^2 T \to \infty$. It can also be supported by the so-called 'principle of detailed balance'—due to microscopic reversibility in thermal equilibrium.

$$g_m B_m^n = g_n B_n^m. \tag{6.38}$$

To simplify our writing we shall from now on absorb the degeneracy factor g into the B coefficient. The spectral density that satisfies this dynamic equilibrium condition (6.37) then becomes

$$\rho = \frac{A_n^m}{B_n^m} \frac{1}{e^{(\epsilon_n - \epsilon_m)/k_B T} - 1} \tag{6.39}$$

which is just Planck's law when we apply the Bohr quantum condition (6.20) together with fixing the coefficient ratio to be

$$\frac{A_n^m}{B_n^m} = \alpha \nu^3. \tag{6.40}$$

The constant α can further be determined, for example, by the Rayleigh–Jeans law. Thus

$$\frac{A_n^m}{B_n^m} = \frac{8\pi \nu^2}{c^3} h\nu. \tag{6.41}$$

Recall that we have used the expression for the radiation density of states (4.1) in the derivation of the Rayleigh–Jeans law.

6.3.2 Stimulated emission and the idea of the laser

Einstein's prediction of stimulated emission became a key element in the invention of the LASER—light amplification by stimulated emission of radiation. Such a device can produce high-intensity collimated coherent electromagnetic waves. In essence a laser is a cavity filled with a "gain medium". We can illustrate the function of this medium by assuming it to be composed of some two-state atoms (e.g. such as the one shown in Fig. 6.3). A positive feedback process, based on stimulated emission, is instituted. The frequency of the input radiation is arranged to match the emission frequency of the medium. This process is amplified by the stimulated emission. If the cavity is enclosed by two mirrors so that light is repeatedly passing back and forth through the gain medium, more and more atoms reside in the excited states (called population inversion) and its intensity can be greatly increased. Because the light originates from a single transition between two fixed levels (unlike an ordinary light source with many different transitions), it is monochromatic with a great deal of coherence. Clearly the invention of the laser required many technical breakthroughs before its realization in the 1950s; nevertheless its basic idea came from the prediction of stimulated emission made by Einstein in 1916.

While stimulated emission can still be understood as a perturbation by an existing field, it would involve a new theoretical framework to understand spontaneous emission. In this case a new photon would have to be created. If it is due to some perturbation, how would the vacuum be the cause? This brings us to the topic of quantum field theory.

6.4 Looking ahead to quantum field theory

At the beginning of this chapter, we discussed the riddle of radiation's wave–particle duality as shown by Einstein's calculation of energy fluctuations (6.16). The Planck's formula for blackbody radiation leads to two terms, one showing the radiation as a system of waves and another as particles. This heightens the apparent contradiction of Einstein's original discovery of thermal radiation (in the Wien limit) behaving thermodynamically like a gas of particles, even though radiation has the familiar wave property of interference, etc. Here we first explain the resolution as provided by the advent of quantum mechanics in 1925–26. The other key property of light quanta that they obey Bose–Einstein statistics will be discussed in Chapter 7.

The new quantum theory is the work of Louis de Broglie, Werner Heisenberg (1901–76), Max Born (1882–70), Pascual Jordan (1902–80), Wolfgang Pauli (1900–58), Erwin Schrödinger, and Paul Dirac (1902–84). In particular electromagnetic radiation is described by a quantized field. This is the subject of quantum electrodynamics. We shall provide, very briefly, some of the basic elements of quantum field theory (QFT).[7] Of course, Einstein never accepted quantum mechanics as a complete theory. His objection to this new quantum theory was mainly in the area of the interpretation of measurement. That will be the topic of our Chapter 8.

[7] A lively and insightful introduction to QFT can be found in Zee (2010).

6.4.1 Oscillators in matrix mechanics

Recall our discussion in Section 3.1 that a radiation field (as a solution to Maxwell's wave equation) can be thought of as a collection of oscillators. Fourier components of waves obey simple harmonic oscillator equations. A quantized radiation field is a collection of quantum oscillators—simple harmonic oscillators as described by quantum mechanics. Quantum field theory is usually presented as the union of quantum mechanics and special relativity. This is so as the Maxwell wave equation satisfies special relativity. When this radiation theory is generalized to other particles, one would work with other relativistic wave equations such as the Dirac equation and Klein–Gordon equation. But the basic features of a quantized field discussed below remain the same.

The essence of quantum mechanics is that physical states are taken to be members of a linear vector space, the Hilbert space, obeying the superposition principle (the basic property of waves), and physical observables are operators represented, for example, by matrices. These observables obey the same dynamical equations as in classical physics, but the kinematics are changed because they, being operators, may no longer be mutually commutative. Thus two operators $\hat{A}$ and $\hat{B}$ may have nonvanishing commutator $\hat{A}\hat{B} - \hat{B}\hat{A} \equiv \left[\hat{A}, \hat{B}\right] \neq 0$. As we shall see, this noncommutivity brings about the particle nature of the system. Planck's constant enters the theory through these commutation relations.

Simple harmonic oscillator Hamiltonian in terms of ladder operators

Here is the quantum mechanical description of a simple harmonic oscillator. The total energy (sum of kinetic and potential energies) is represented by the Hamiltonian operator, which can be expressed in terms of the position and momentum operators $(\hat{x}, \hat{p})$. With the angular frequency ω, the Hamiltonian is given by

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2. \tag{6.42}$$

The momentum and position operators are postulated to satisfy the 'canonical' commutation relation

$$\left[\hat{x}, \hat{p}\right] = i\hbar. \tag{6.43}$$

We can factorized the oscillator Hamiltonian, in terms of the **ladder operators**:[8]

$$\hat{a}_{\pm} = \frac{1}{\sqrt{2m}}(\mp i\hat{p} + m\omega\hat{x}). \tag{6.44}$$

A simple calculation shows that they have the product relations

$$\hat{a}_+\hat{a}_- = \hat{H} + \frac{i\omega}{2}\left[\hat{x}, \hat{p}\right] \quad \text{and} \quad \hat{a}_-\hat{a}_+ = \hat{H} - \frac{i\omega}{2}\left[\hat{x}, \hat{p}\right]. \tag{6.45}$$

From the sum and difference of these two expressions, we obtain the Hamiltonian

$$\hat{H} = \frac{1}{2}(\hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+) \tag{6.46}$$

and the commutator of the ladder operators, which is just a simple transcription of (6.43),

$$\left[\hat{a}_{\mp}, \hat{a}_{\pm}\right] = \pm\hbar\omega. \tag{6.47}$$

[8] Since $\hat{x}$ and $\hat{p}$ are Hermitian operators, these ladder operators are each other's Hermitian conjugates, $\hat{a}_+^\dagger = \hat{a}_-$ and $\hat{a}_-^\dagger = \hat{a}_+$.

An application of this commutation relation to (6.46) leads to

$$\hat{H} = \hat{a}_\mp \hat{a}_\pm \mp \frac{1}{2}\hbar\omega. \tag{6.48}$$

Raising and lowering energy levels by $\hat{a}_\pm$

Consider the states $\hat{a}_\pm |n\rangle$, obtained by applying the ladder operators $\hat{a}_\pm$ to an energy eigenstate $|n\rangle$ with $\hat{H}|n\rangle = E_n |n\rangle$. To find the energy of such states, we probe them by the Hamiltonian operator in the form of (6.48):

$$\hat{H}\hat{a}_\pm |n\rangle = \left(\hat{a}_\mp \hat{a}_\pm \hat{a}_\pm \mp \frac{1}{2}\hbar\omega \hat{a}_\pm\right)|n\rangle\,. \tag{6.49}$$

Since (6.47) implies the commutation relation

$$\left[\hat{a}_\mp \hat{a}_\pm, \hat{a}_\pm\right] = \left[\hat{a}_\mp, \hat{a}_\pm\right]\hat{a}_\pm + \hat{a}_\mp\left[\hat{a}_\pm, \hat{a}_\pm\right] = \pm\hbar\omega\hat{a}_\pm + 0, \tag{6.50}$$

the RHS of (6.49), after interchanging the order of $\hat{a}_\mp \hat{a}_\pm$ and $\hat{a}_\pm$ by the commutator (6.50), becomes

$$\hat{H}\hat{a}_\pm |n\rangle = \left(\hat{a}_\pm \hat{a}_\mp \hat{a}_\pm \pm \hbar\omega\hat{a}_\pm \mp \frac{1}{2}\hbar\omega \hat{a}_\pm\right)|n\rangle\,.$$

We can factor out $\hat{a}_\pm$ to the left and use the expression of $\hat{H}$ as given in (6.48) to have

$$\begin{aligned}\hat{H}\left(\hat{a}_\pm |n\rangle\right) &= \hat{a}_\pm\left(\hat{H} \pm \hbar\omega\right)|n\rangle \\ &= \hat{a}_\pm\left(E_n \pm \hbar\omega\right)|n\rangle = \left(E_n \pm \hbar\omega\right)\left(\hat{a}_\pm |n\rangle\right).\end{aligned} \tag{6.51}$$

This calculation shows that the states $\hat{a}_\pm |n\rangle$ are also eigenstates of the Hamiltonian with energy values $E_n \pm \hbar\omega$. This explains why $\hat{a}_+$ is called the **raising operator** and $\hat{a}_-$ the **lowering operator**.

The quantized energy spectrum derived

Just like the classical oscillator case, the energy must be bounded below. We denote this lowest energy state, the ground state, by $|0\rangle$. Since the ground state cannot be lowered further, we must have the condition:

$$\hat{a}_- |0\rangle = 0. \tag{6.52}$$

From this we deduce that the ground state energy E_0 does not vanish:

$$\hat{H}|0\rangle = E_0 |0\rangle = \left(\hat{a}_+ \hat{a}_- + \frac{1}{2}\hbar\omega\right)|0\rangle = \frac{1}{2}\hbar\omega\,|0\rangle\,. \tag{6.53}$$

Namely $E_0 = \frac{1}{2}\hbar\omega$, which is often referred to as the **zero-point energy**. On the other hand, all the excited states can be reached by the repeated application of the raising operator[9] to the ground state:

$$\left(\hat{a}_+\right)^n |0\rangle \sim |n\rangle\,. \tag{6.54}$$

[9]The proportionality constants will be worked out below when we discuss the normalization of quantum states.

According to (6.51), each application of $\hat{a}_+$ raises the energy by one $\hbar\omega$ unit: we thus derive the energy of a general state $|n\rangle$:

$$\hat{H}\,|n\rangle = E_n\,|n\rangle = \left(n + \frac{1}{2}\right)\hbar\omega\,|n\rangle \tag{6.55}$$

with $n = 0, 1, 2, \ldots$ and

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega. \tag{6.56}$$

This result agrees, up to a constant of $\hbar\omega/2$, with the Planck oscillator energy quantization proposal. Before we move on to quantum field theory we note two technicalities of the quantum theory of oscillators.

The zero-point energy A comparison of (6.46) and (6.48) with (6.56) shows clearly that the zero-point energy, $E_0 = \frac{1}{2}\hbar\omega$, originates from the noncommutivity of the position with momentum, or equivalently the $\hbar\omega$ factor in the commutator (6.47). Physically one can understand the presence of this ground state energy by the uncertainty principle. Even in the absence of any quanta, an oscillator still has the natural length-scale of $x_0 = \sqrt{\hbar/m\omega}$; thus, the **uncertainly principle**[10] for the position and momentum observables, $\Delta x \Delta p \gtrsim \hbar$, requires a minium momentum of $p_0 = \sqrt{m\hbar\omega}$. This translates into a minium energy of $E_0 = p_0^2/2m = \frac{1}{2}\hbar\omega$—just the zero-point energy.

[10]The uncertainty relation is a direct mathematical consequance of the noncommutivity of observables.

The number operator and the normalization of oscillator states A simple comparison of (6.55) with (6.48) suggest that we can define a 'number operator' $\hat{n} \equiv \hat{a}_+\hat{a}_-/(\hbar\omega)$ so that $\hat{H} = (\hat{n} + \frac{1}{2})\hbar\omega$ and $\hat{n}\,|n\rangle = n\,|n\rangle$. This operator is Hermitian $\hat{n}^\dagger = \hat{a}_-^\dagger \hat{a}_+^\dagger/(\hbar\omega) = \hat{n}$, with real eigenvalues $n = 0, 1, 2, \ldots$. All quantum mechanical states must be normalized (as they have the interpretation of a probability): $\langle n\,|n\rangle = ||n\rangle|^2 = 1$, and $\langle n-1\,|n-1\rangle = ||n-1\rangle|^2 = 1$, etc. From these we can find out how the ladder operators act on the number states $\hat{a}_-\,|n\rangle = c\,|n-1\rangle$ with the coefficient c determined as follows. Starting with

$$\langle n|\,\frac{\hat{a}_+\hat{a}_-}{\hbar\omega}\,|n\rangle = \langle n|\,\hat{n}\,|n\rangle = n, \tag{6.57}$$

we have, using the hermiticity properties $\hat{a}_\pm^\dagger = \hat{a}_\mp$,

$$n\hbar\omega = \langle n|\,\hat{a}_+\hat{a}_-\,|n\rangle = |\hat{a}_-\,|n\rangle|^2 = |c|^2\,||n-1\rangle|^2 = |c|^2\,,$$

hence $c = \sqrt{n\hbar\omega}$. Similarly we can work out the effects of $\hat{a}_+$. Thus the effects of the ladder operators are

$$\hat{a}_-\,|n\rangle = \sqrt{n\hbar\omega}\,|n-1\rangle \quad \text{and} \quad \hat{a}_+\,|n\rangle = \sqrt{(n+1)\,\hbar\omega}\,|n+1\rangle\,. \tag{6.58}$$

6.4.2 Quantum jumps: From emission and absorption of radiation to creation and annihilation of particles

A quantum radiation field is a collection of quantum oscillators. The energy spectrum of the field for each mode is given by the quantized energy as shown in (6.56). Thus the Planck/Einstein quantization result is automatically obtained in the framework of quantum field theory. The first application of the new quantum mechanics to the electromagnetic field was given in the famous three-man paper (*Dreimännerarbeit*) of Born, Heisenberg, and Jordan (1926). This had at last put Einstein's idea of the photon on a firm mathematical foundation. The new quantum mechanics also yields the correct hydrogen spectrum, as shown by Pauli (1926) in matrix mechanics and by Schrödinger (1926) in wave mechanics.

Vacuum energy fluctuation

The new feature of (6.56) is the presence of the zero-point energy. Since the ground state of a field system is identified with the vacuum, quantum field theory predicts a nonvanishing energy for the vacuum state. While the presence of this constant energy term would not affect quantum applications such as the photoelectric effect and specific heat, as we shall see, there are observable effects associated with this nonvanishing vacuum energy. In fact what we have is the fluctuation of the energy in the vacuum state. We have already discussed the zero-point oscillator energy from the viewpoint of the position–momentum uncertainty relation. We also have the uncertainty relation[11] between energy and time, $\Delta E \Delta t \gtrsim \hbar$. This suggest that, for a sufficiently short time interval, energy can fluctuate, even violating energy conservation. The vacuum energy is the (root-mean-square) average of the fluctuation energy.

[11] Time is not a dynamical observable represented by an operator in quantum mechanics. The uncertainty relation follows from the Heisenberg equation of motion with Δt being the characteristic time that a system takes to change.

Emission and absorption of radiation

That the formalism of the quantum oscillator allows one to raise and lower the field energy by units of $\hbar\omega$ can naturally be used to describe the quantum jumps of emission and absorption of radiation. In particular, the amplitude for the emission (i.e. creation) of an energy quantum is directly related to the matrix element:

$$\langle n+1|\,\hat{a}_+\,|n\rangle = \sqrt{(n+1)\,\hbar\omega}. \tag{6.59}$$

The equality follows from (6.58), leading to an emission rate proportional to the factor $(n+1)$.

This is just the result first discovered by Einstein. From the RHS of (6.36) we have the total (induced and spontaneous) emission rate,

$$\left(\frac{dP_n}{dt}\right)_{\text{em}} = \left(\rho\frac{B_n^m}{A_n^m}+1\right)A_n^m P_n = \left(\frac{\rho c^3}{8\pi h\nu^3}+1\right)A_n^m P_n, \tag{6.60}$$

where we have used the relation for Einstein's A and B coefficients (6.41). The language of quantum field theory allows us to express this emission rate in terms of the **number of light quanta** n:

$$\left(\frac{dP}{dt}\right)_{\text{em}} \propto (n+1)\,P, \tag{6.61}$$

because, according to Eq. (4.1), we have the energy (per radiation oscillator) $U = \rho c^3/8\pi\nu^2$ and $U/h\nu = n$. While the factor of n on the RHS of (6.61) corresponds to the stimulated emission, the factor of 1 in $(n+1)$ reflects the spontaneous emission. Thus in quantum oscillator language, the spontaneous emission term comes from the commutation relation (6.47). It has exactly the same origin as the zero-point energy. This linkage between the vacuum energy and spontaneous emission suggests to us that we can identify spontaneous emission as brought about (i.e. due to the perturbation) by the vacuum energy fluctuation.

Creation and annihilation of particles

In the above we have seen that the raising and lowering ladder operators of the oscillator provide us with a natural description of emission and absorption of radiation in units of the energy quanta. In modern language this is the emission and absorption of photons.

Even with the success of the quantum field theory treatment of radiation, we still have, at this stage, a dichotomy: on one hand we have radiation with its quanta that can be freely created and destroyed; on the other hand, material particles such as electrons and protons were thought to be eternal. Further development of quantum field theory showed that material particles can also be thought of as quanta of various fields, in just the same way that the photon is the quantum of the electromagnetic field. These matter fields are also collections of their oscillators, with their corresponding ladder operators identified as the creation and annihilation operators of these material particles.

This is a major advance in our understanding of particle interactions. Until then the interactions among particles were described by forces that can change the motion of particles. Photons are just like other particles except they have zero rest-mass. While there is no energy threshold for radiation, given enough energy all particles can appear and disappear through interactions. The first successful application of this idea was in the area of nuclear beta decay. The nucleus is composed of protons and neutrons. How is it then possible for one parent nucleus to emit an electron (and a neutrino) while changing into a different daughter nucleus? Enrico Fermi (1901–54) gave the quantum field theoretical answer to this puzzle. He modeled his theory of beta decay on quantum electrodynamics and described the process as the annihilation of a neutron in the parent nucleus followed by the creation of a proton in the final state nucleus along with the creation of the electron (and the neutrino).

Ranges of interactions

In a field theory the interaction between the source particle and test particle is described as the source particle giving rise to a field propagating out from the source and the field then acting locally on the test particle. Since a quantized field can be thought of as a collection of particles, this interaction is depicted as an exchange of particles between the source and test particles. Since the exchanged particle can have a mass, the creation of such an exchange particle (from the vacuum) would involve an energy nonconservation of $\Delta E \simeq mc^2$. But the uncertainty principle $\Delta E \Delta t \geq \hbar$ only allows this to happen for a time interval of $\Delta t \simeq \hbar/mc^2$. This implies a propagation, hence an interaction range, of $R \simeq c\Delta t \simeq \hbar/mc$ (the Compton wavelength of the exchanged particle). Electromagnetic interaction is long range because the photon is massless. Based on such considerations, Hideki Yukawa (1907–81) predicted the existence of a meson, about a couple of hundred times more massive than the electron, as the mediating quantum of nuclear forces which were known to have a finite range of about a fermi ($= 10^{-15}$ m).

We conclude by noting the central point of quantum field theory: The essential reality is a set of fields, subject to the rules of quantum mechanics and special relativity; all else is derived as a consequence of the quantum dynamics of these fields (Weinberg 1977).

6.4.3 Resolving the riddle of wave–particle duality in radiation fluctuation

In this last section we return to the issue of wave–particle duality displayed by the radiation energy fluctuation discussed at the beginning of this chapter. How does quantum field theory resolve the riddle of the radiation fluctuation having two factors (6.16): a wave term plus a particle term?

In quantum field theory a field is taken to be an operator. The above discussion of the radiation field being a collection of quantum oscillators means the replacement of a classical field (a complex number) $Ae^{i\phi_j}$, with appropriate normalization, by an operator $\hat{a}_{j-}e^{i\phi_j} + \hat{a}_{j+}e^{-i\phi_j}$ with $\left[\hat{a}_{j-}, \hat{a}_{k+}\right] = \hbar\omega\delta_{jk}$. The calculation of the energy fluctuation of such a wave system follows the same lines as that for classical waves (cf. SuppMat Section 6.5). However the non-commutivity of quantum oscillator operators $\hat{a}_{\pm}$ gives rise to extra terms, as shown in (6.47). The result is that, instead of the classical wave result of (6.8), we now have the mean-square energy density

$$\Delta u^2 = u^2 + u\hbar\omega. \tag{6.62}$$

For the system average we follow the same procedure used in Section 6.1, to obtain $\langle E\rangle = Nd\nu u$ and $\left\langle \Delta E^2\right\rangle = Nd\nu\Delta u^2$, and, using the density of states result $N = 8\pi\nu^3/c^3$ of (4.5), to arrive at the final result of

$$\left\langle \Delta E^2\right\rangle = \frac{\langle E\rangle^2}{Nd\nu} + \langle E\rangle\,\hbar\omega = \frac{c^3}{8\pi\nu^2}\frac{\langle E\rangle^2}{\nu d\nu} + \langle E\rangle\,h\nu. \tag{6.63}$$

This is just the result (6.16) that Einstein obtained in 1909 from Planck's distribution. Thus these two terms, one wave and one particle, can be explained in a unified framework. Recall that it was based on this result that Einstein first proposed the point-like particles as the quanta of radiation. Alas, as already mentioned above, Einstein never accepted this beautiful resolution of the great wave–particle riddle, as he never accepted the framework of the new quantum mechanics.

The extra particle-like term comes from the commutator (6.47) which is equivalent to $\left[\hat{x}, \hat{p}\right] = i\hbar$. Thus it has the same origin as the zero-point energy and the energy quantization feature of the quantized wave system. This elegant resolution of the wave–particle duality was discovered by Pascual Jordan (Born, Heisenberg, and Jordan 1926).[12] Somehow this result is not well-known generally; the full story of, and a careful re-derivation of, Jordan's contribution was given by Duncan and Janssen (2008).

[12] There is ample historical evidence showing that Jordan was alone responsible for this section of the *Dreimännerarbeit.*

Quantum field theory can account for another fundamental feature of a system of many particles: its quantum statistics property. As we shall discuss in the next chapter, photons obey Bose–Einstein statistics and the new quantum mechanics requires its state to be symmetric under the exchange of any two photons. It turns out that the commutation relation that is being discussed in this section [cf. Eq. (6.47)] is just the elegant mathematical device needed to bring about this required symmetry.[13] This quantum statistical property leads directly to the Planck distribution for a thermal photon system. Planck's distribution yields a fluctuation showing the wave–particle duality. Thus quantum field theory gives a completely self-consistent description of the electromagnetic radiation. In this theory one can see the effects of waves and particles simultaneously.

[13] For a fermionic system, the particle creation and annihilation operators are postulated to obey anticommutation relations $\hat{a}_{j-}\hat{a}_{k+} + \hat{a}_{k+}\hat{a}_{j-} \equiv \{\hat{a}_{j-}, \hat{a}_{k+}\} = \hbar\omega\delta_{jk}$ so that a multi-fermion system is antisymmetric under the interchange of two identical fermions.

6.5 SuppMat: Fluctuations of a wave system

Here is a calculation of the fluctuations of randomly superposed waves. This presentation follows that given by Longair (2003, p. 369). The energy density is proportional to the field squared $|F|^2$. For the case of electromagnetic waves, F can be the electric or magnetic field. We assume that all polarization vectors are pointing in the same direction, reducing the problem to a scalar field case, and all waves have the same amplitude A. [Cf. Eq. (3.4) in Section 3.1] In this way, we have the energy density as

$$|F|^2 = A^2 \left(\sum_{j=1}^{N} e^{i\phi_j} \right)^* \left(\sum_{k=1}^{N} e^{i\phi_k} \right) = A^2 \left(N + \sum_{j \neq k} e^{i(\phi_k - \phi_j)} \right). \tag{6.64}$$

When the phases of the waves are random, the second term in the parentheses being just sines and cosines, averages out to zero:

$$u = \left\langle |F|^2 \right\rangle = NA^2. \tag{6.65}$$

Namely, the total average energy density of a set of incoherent waves is simply the sum of the energy density of each mode.

To calculate the mean-squared energy, we need to calculate the square of the energy density (6.64). The result is

$$\begin{aligned} \left| |F|^2 \right|^2 &= A^4 \left| \left(N + \sum_{j \neq k} e^{i(\phi_k - \phi_j)} \right) \right|^2 \\ &= A^4 \left(N^2 + 2N \sum_{j \neq k} e^{i(\phi_k - \phi_j)} + \sum_{j \neq k} e^{-i(\phi_k - \phi_j)} \sum_{l \neq m} e^{i(\phi_m - \phi_l)} \right). \end{aligned}$$

Again the second term (with coefficient $2N$), as well as most of the terms in the double sum, average out to zero. The terms in the double sum that survive are those with matching indices $j = l$ and $k = m$; there are thus N^2 such terms. The result then is

$$\left\langle \left| |F|^2 \right|^2 \right\rangle = A^4 \left(N^2 + 0 + N^2 \right) = 2N^2 A^4. \tag{6.66}$$

We have the variance of the fluctuating wave energy:

$$\Delta u^2 = \left\langle \left| |F|^2 \right|^2 \right\rangle - \left\langle |F|^2 \right\rangle^2 = N^2 A^4 = u^2. \tag{6.67}$$

This is the claimed result for wave fluctuations as displayed in (6.8).