

Preface

Einstein explained in equations

Albert Einstein's achievement in physics is proverbial. Many regard him as the greatest physicist since Newton. What did he do in physics that's so important? While there have been many books about Einstein, most of these explain his achievement only in qualitative terms. This is rather unsatisfactory as the language of physics is mathematics. One needs to know the equations in order to understand Einstein's physics: the precise nature of his contribution, its context, and its influence. The most important scientific biography of Einstein has been the one by Abraham Pais: *Subtle is the Lord... The Science and the Life of Albert Einstein*: The physics is discussed in depth; however, it is still a narrative account and the equations are not worked out in detail. Thus this biography assumes in effect a high level of physics background that is perhaps beyond what many readers, even working physicists, possess. Our purpose is to provide an introduction to Einstein's physics at a level accessible to an undergraduate physics student. All physics equations are worked out from the beginning. Although the book is written with primarily a physics readership in mind, enough pedagogical support material is provided that anyone with a solid background in an introductory physics course (say, an engineer) can, with some effort, understand a good part of this presentation.

In historical context This is a physics book with material presented in the historical context. Although it is not a scholarly history and there is hardly any original work in the Einstein biography, historical material from secondary sources is used to make the physics material more comprehensible and interesting. For example, a more careful discussion of the results obtained by Hendrik Lorentz will precede Einstein's special relativity. Planck's and Einstein's work on blackbody radiation are presented only after reviewing first the thermodynamics and scaling results of Wilhelm Wien. Our opinion is that the history conveyed through standard physics textbooks sometimes misses the proper context of the discovery. The original Einstein story is actually more interesting and illuminating.

Post-Einstein development Also, we do not stop at Einstein's discovery, but carry the discussion onto some of the advances in physics that had been made because of Einstein's contribution. We discuss gauge symmetry leading to the Standard Model of particle physics as a legacy of Einstein's invariance-principle approach. As an example of Einstein's unified field theory we present the Kaluza–Klein unification of electromagnetism and gravitation in a space

with an extra dimension. Such knowledge is needed to fully appreciate the profound influence that Einstein's physics had on subsequent development.

Can you answer these “Einstein questions”?

Physics students have already learnt aspects of Einstein's physics—from segments in their course work or from popular accounts. Here is a list of 21 Einstein questions. Can you answer them?¹

¹Brief answers are given in Appendix C, where the reader can also find the chapter and section numbers where the discussion of, and answer to, such Einstein questions are carried out in the text.

1. Einstein's research played a significant part in people's acceptance of the reality of the molecular constituents of matter. In one year, 1905, he showed three separate ways to deduce Avogadro's number from macroscopic measurements. What were the three areas in physics where these deductions were made? Surprisingly, one of these was the blackbody radiation.
2. Einstein's celebrated Brownian motion paper did not have the words “Brownian motion” in its title. How come?
3. Einstein's 1905 photoelectric paper, where the idea of light quanta was first proposed, and which was the work cited when he was awarded the Nobel Prize, was concerned mostly with a statistical study of blackbody radiation. If the papers on quantum theory by Planck and by Einstein were both concerned with blackbody radiation, what was their key difference?
4. In the classical theory we have an “ultraviolet catastrophe” for the blackbody radiation. How does the postulate of energy quantization cure this problem?
5. Einstein's quantum theory of specific heat is historically important because it is the first instance when the quantum idea was shown to be relevant to physical systems well beyond the esoteric case of blackbody radiation. His theory also clarified the questions about matter's molecular composition. How is that so?
6. The statement of wave–particle duality was made first by Einstein in his 1909 study of fluctuations of radiation energy. Einstein and Bohr had influenced each other's work, especially with respect to the idea of quantum transitions (the quantum jumps). How did quantum mechanics and quantum field theory accommodate, in one elegant framework, simultaneously waves, particles, and quantum jumps? Famously, this is not the resolution that Einstein was able to accept.
7. Einstein never accepted the orthodox interpretation of quantum mechanics. Was he just too set in his ways to appreciate the new advances in physics? How had Einstein's criticism influenced subsequent investigation on the meaning of quantum mechanics?
8. By the time Einstein proposed his special theory of relativity, the Lorentz transformation had already been written down. Einstein was unaware of this latest development, as he was working (in the Swiss Patent Office) outside an academic environment. Einstein's derivation of this transformation rule differed fundamentally from the way it was gotten by Lorentz and others. How?
9. While the Michelson–Morley measurement did not play a direct role in Einstein's motivation for special relativity, there were other results

(stellar aberration, Fizeau's experiment, and Fresnel's formula) that Einstein had acknowledged as having had an influence. In what ways were they relevant to Einstein's motivation? How were they explained in the final relativity theory?

10. The key element of special relativity is the new conception of time. Just about all the counter-intuitive relativistic effects spring from the relativity of simultaneity. What about the well-known result of "length contraction"? Does it have this connection with time also? If so, how?
11. What is the difference between the special and general theories of relativity? Special relativity is applicable to electromagnetism, mechanics, thermodynamics, etc. but not to gravity (why not?); on the other hand, general relativity is the field theory of gravitation. Why then is special relativity special and general relativity general? Why does the principle of general relativity automatically bring gravity into the consideration?
12. Einstein originally dismissed Minkowski's geometrical formulation of his relativity theory as "superfluous learnedness". What caused Einstein to change this appraisal later on? With respect to the role of mathematics in the discovery of physics theory, how did Einstein's view evolve? Was Einstein a great mathematician as well as a great physicist? What difference would it make?
13. What was the idea that Einstein called "my happiest thought"? Which moment of elation was characterized by his biographer Pais as "by far the strongest emotional experience in Einstein's scientific life, perhaps, in all his life"?
14. Einstein's general relativity is said to be a geometric theory of gravity. What does one mean by a "geometry theory"? How did Einstein get the idea that "a gravitational field is simply spacetime with curvature"? To what physical realm exactly does Einstein's theory extend Newtonian gravity?
15. One way to state the equivalence of inertia and gravitation is to say that gravity can always be transformed away locally (by going to a reference frame in free fall). Thus the essence of gravity is represented by its differentials (tidal forces). How does this feature appear in the field equation of general relativity (the Einstein equation)?
16. According to Einstein's gravity theory, shaking a mass gives rise to gravitational waves. Because gravity is such a weak force, it is extremely difficult to detect the resultant waves. At this moment there is no confirmed evidence for their direct detection. Nevertheless, consequences of gravitational wave emission have been measured and found in agreement with Einstein's prediction. What are these indirect effects?
17. It is well known that black holes are regions where gravity is so strong even light cannot escape. Black holes demonstrate the full power and glory of general relativity also because, inside black holes, "the roles of space and time are interchanged"! What does this mean? How does this come about?
18. Planck's discovery of Planck's constant allowed him to construct, from h , c , and G_N , a natural unit system of mass-length-time. Through the essential contribution by Einstein, we now understand each of these

fundamental constants as the “conversion factor” that connects disparate realms of physics. Can you name these areas? What are Einstein’s works that made these syntheses possible?

19. The modern study of cosmology started with Einstein’s 1917 paper. The story was often told that Einstein regarded his introduction of the cosmological constant as “the biggest blunder of my life”. What is the source of this piece of anecdotal history? What role does Einstein’s cosmological constant play in our present understanding of the universe?
20. Special relativity, photons, and Bose–Einstein statistics are crucial ingredients of modern particle physics. On the other hand, Einstein did not work directly on any particle theory. Yet, one can still claim that the influence of his ideas had been of paramount importance in the successful creation of the Standard Model of particle physics. What is the basis of this claim?
21. In the later years of his life, Einstein devoted the major part of his physics effort in the search for a unified field theory. Was this just a misguided chasing of an impossible dream? Based on our current understanding, what was the legacy of this somewhat less appreciated part of his research?

Clearly the story is a fascinating one. But to understand it properly one needs to know the relevant physics, to know some of the technical details. This, an undergraduate physics student, with some help, should be able to do.

Atoms, quanta, and relativity—Our presentation

The material is logically divided into five parts: atoms, quanta, special/general relativity, and later developments. Each of the 17 chapters has a detailed summary, in the form of a bullet list, placed at the beginning of the chapter. The reader can use these lists to get an overview of the contents and decide which part the book he or she wants to study in detail. For example, a reader may well wish to postpone Chapter 1 for a later reading; it discusses Einstein’s doctoral thesis and concerns the subject of classical hydrodynamics, which may not be all that familiar to a present-day student.

Physics focus Although many of Einstein’s papers are discussed in this book, his physics is not presented in the exact form as given in his papers. For example, the derivation of the Lorentz transformation is different from that given in Einstein’s 1905 paper, even though the assumption and result are the same. In finding the general relativity field equation, Einstein’s original steps are not followed because, after Einstein’s discovery, it had been shown by others that the same conservation law condition could be obtained much more simply by using the Bianchi identities. In other words, the focus of this book is Einstein’s physics, rather than the strict historical details of his physics. It is hoped that our presentation (without the obsolete notation of the original papers) is more accessible to a modern-day reader.

As a textbook? Since Einstein’s legacy has permeated so many areas in physics, a wide range of topics will be covered in our presentation. It is hoped that after studying these lessons, a student will not only have learnt

some history of physics and a better appreciation of Einstein's achievement, but, perhaps more importantly, will have enhanced their understanding of some of the basic areas in their physics curriculum (and a glimpse of more advanced topics): thermodynamics, hydrodynamics, statistical mechanics, Maxwell's equations, special and general relativity, cosmology, quantum mechanics, quantum field theory, and particle physics. Although this book is written for a general-interest physics readership, it can be used as a textbook as well—for a "Special Topics" course, or an "Independent Reading" course. One possibility is to have the book function as the basis of a "senior year project". Working through the book may well be an enjoyable experience for both the student and the instructor.

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Book website As always, I shall be glad to receive, at tpcheng@umsl.edu, readers' comments and notification of errors. An updated list of corrections will be displayed at <http://www.umsl.edu/~tpcheng/einstein.html>.

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T.P.C.

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