Solutions to selected problems

2.2 **Inverse Lorentz transformation** The Lorentz transformation (2.46), and its inverse, written out only for the nontrivial components are

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \gamma \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix}. \tag{1}$$

The inverse matrix relation is demonstrated, using $\gamma^2 = (1-\beta^2)^{-1}$, by

$$\gamma^2 \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

2.3 **Lorentz transformation of derivative operators**

(a) Start with the chain rule,

$$\frac{\partial}{\partial x'} = \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial x'}\frac{\partial}{\partial t} = \gamma\frac{\partial}{\partial x} + \gamma\beta\frac{\partial}{c\partial t}.$$

To reach the last equality, we used (1) showing (ct, x) as functions of (ct', x') to calculate $\partial x/\partial x' = \gamma$ and $\partial t/\partial x' = \gamma\beta/c$. Similarly, we have

$$\frac{\partial}{c\partial t'} = \gamma\frac{\partial}{c\partial t} + \gamma\beta\frac{\partial}{\partial x}.$$

(b) $\bar{\mathbf{L}}$ can be found by substituting into $\delta^\nu_\mu = \partial(x'_\nu)/\partial x'_\mu \equiv \partial'_\mu x'^\nu$ the respective Lorentz transformations $\mathbf{L}$ and $\bar{\mathbf{L}}$ for coordinates and coordinate derivatives Eqs. (2.47) and (2.48):

$$\begin{aligned}\delta^\nu_\mu = \partial'_\mu x'^\nu &= \sum_{\lambda,\rho}\left(\bar{\mathbf{L}}_\mu^{\ \lambda}\partial_\lambda\right)\left(\mathbf{L}^\nu_{\ \rho}x^\rho\right) \\ &= \sum_{\lambda,\rho}\bar{\mathbf{L}}_\mu^{\ \lambda}\mathbf{L}^\nu_{\ \rho}\delta^\rho_\lambda = \sum_\lambda \bar{\mathbf{L}}_\mu^{\ \lambda}\mathbf{L}^\nu_{\ \lambda}.\end{aligned} \tag{2}$$

Namely, $\mathbf{1} = \bar{\mathbf{L}}\mathbf{L}$. Thus, the transformation for the coordinate derivative operators is just the inverse shown in (1)—as already indicated by the substitution of $v \to -v$.

2.4 **Lorentz covariance of Maxwell's equations** Given (2.51), we show the validity of (2.50) by applying the Lorentz transformations for the fields and spacetime derivatives:

$$\nabla' \cdot \mathbf{B}' = \frac{\partial B'_x}{\partial x'} + \frac{\partial B'_y}{\partial y'} + \frac{\partial B'_z}{\partial z'}$$

$$= \gamma\left(\frac{\partial}{\partial x} + \beta\frac{\partial}{c\partial t}\right)B_x + \frac{\partial}{\partial y}\gamma\left(B_y + \beta E_z\right) + \frac{\partial}{\partial z}\gamma\left(B_z - \beta E_y\right)$$

$$= \gamma\underbrace{\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right)}_{\nabla\cdot\mathbf{B}=0} + \gamma\beta\underbrace{\left[\frac{\partial B_x}{c\partial t} + \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\right]}_{\left(\nabla\times\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}\right)_x=0} \qquad (3)$$

where we have used Lorentz transformation of (2.49) and (2.16) to reach the second line. The x-component of Faraday's equation is singled out because we have assumed a Lorentz boost in the x direction.

2.5 **From Coulomb's to Ampere's law** We illustrate the general approach by the example of a derivation of Faraday's law from the magnetic Gauss's law. We note that since the magnetic Gauss's law is valid in both frames, $\nabla \cdot \mathbf{B} = 0$ and $\nabla' \cdot \mathbf{B}' = 0$. Eq. (3) implies that the x component of $\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$ is zero. Hence, all three components of $\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0$ (all of the vector components are zero) as the y and z components can be similarly deduced by considering Lorentz boosts in the y and z directions.

2.6 **Length contraction and light-pulse clock** In the rest frame of the clock, the total time $\triangle t'$ for a light pulse to go from one end to another and back is the sum $\triangle t' = \triangle t'_1 + \triangle t'_2$, where $\triangle t'_2$ is the time for the pulse to make the return trip. Clearly $\triangle t'_1 = \triangle t'_2 = L'/c$, where L' is the rest frame length of this clock. Now consider the clock in motion, moving with velocity v from left to right. The length the pulse must travel is lengthened when going from left to right, and shortened when going from right to left (on the return trip), due to the fact that the ends of the light clock are moving to the right:

$$c \triangle t_1 = L + v \triangle t_1, \quad c \triangle t_2 = L - v \triangle t_2,$$

where L and $\triangle t$ are the length and time measured in the moving frame. We can solve the above equations for $\triangle t_1$ and $\triangle t_2$ to get the time it takes the light pulse to go from one end of the light clock to the the other in the moving frame:

$$\triangle t = \triangle t_1 + \triangle t_2 = \frac{L}{c-v} + \frac{L}{c+v} = \gamma^2\frac{2L}{c}. \qquad (4)$$

Using the time-dilation formula (2.26), we can find $\triangle t$ in terms of L'. By equating $\triangle t = \gamma \triangle t' = \gamma\,(2L/c)$ to the result of (4), which gives $\triangle t$ in terms of L, we obtain the Lorentz length-contraction formula of $L = L'/\gamma$.

2.8 **Invariant spacetime interval and relativity of simultaneity**

(a) The invariant spacetime interval gives $-c^2\Delta t'^2 + \Delta x'^2 = \Delta x^2$, or $c\Delta t' = \sqrt{\Delta x'^2 - \Delta x^2}$.

(b) The Lorentz transformation for the spatial coordinates, with $\Delta t = 0$, is $\Delta x' = \gamma\Delta x$. This implies that $\gamma = (1-\beta^2)^{-1/2} = (\Delta x'/\Delta x)$ and $\gamma\beta = \sqrt{\gamma^2 - 1} = \sqrt{(\Delta x'/\Delta x)^2 - 1}$. The Lorentz transformation for the time coordinates then leads to the same result as in (a), $c\Delta t' = \gamma\beta\Delta x = \sqrt{\Delta x'^2 - \Delta x^2}$.

2.9 More simultaneity calculations

(a) Given the Lorentz transformation (2.34) and (2.36), as well as its inverse (2.37) and (2.38), it is clear that $\Delta t' = 0$ implies, through (2.36), $\Delta t = (\beta/c)\Delta x$, and through (2.38), $\Delta t = (\beta/c)\gamma\Delta x'$. These two equalities require the consistency condition $\Delta x = \gamma\Delta x'$, which is compatible with the Lorentz transformation (2.37) with $\Delta t' = 0$.

(b) Our derivation of length contraction in Section 2.2.3 would lead us to expect the result of $\Delta x' = \gamma^{-1}\Delta x$ because the key input of the two ends of an object being measured at the same time in the "moving frame" is satisfied by our $\Delta t' = 0$ condition.

(c) In Section 2.2.2, especially Eqs. (2.24) and (2.25), we have shown that the time intervals for the light signals to reach the back and front ends of the railcar as recorded by the platform observer are

$$t_1 = \frac{L}{2c}\frac{1}{1+\beta}, \quad t_2 = \frac{L}{2c}\frac{1}{1-\beta},$$

where L is the length of the moving railcar as seen by the platform observer. If we let the railcar length as seen by the railcar observer be $\Delta x'$, then the railcar length as seen by the platform observer should be $L = \gamma^{-1}\Delta x'$ due to length contraction. We can calculate the time difference, as in (2.25), to be

$$\Delta t = t_2 - t_1 = \frac{\beta}{c}\gamma^2 L = \frac{\beta}{c}\gamma\Delta x',$$

which agrees with $\Delta t = (\beta/c)\gamma\Delta x'$ obtained above. With respect to the O observer, the emission points are located at

$$x_1 = -ct_1 = -\frac{\Delta x'}{2\gamma}\frac{1}{1+\beta}, \quad x_2 = ct_2 = \frac{\Delta x'}{2\gamma}\frac{1}{1-\beta}.$$

Hence, according to the platform observer, the two emission events have a separation of

$$\Delta x = x_2 - x_1 = \frac{\Delta x'}{2\gamma}\left(\frac{1}{1-\beta} + \frac{1}{1+\beta}\right) = \gamma\Delta x'$$

which agrees with the result $\Delta x = \gamma\Delta x'$ gotten from Lorentz transformation above.

2.10 Reciprocity of twin-paradox measurements In this reciprocal arrangement, the γ factors are exactly the same. Al's yearly flashes are received every 3 years by Bill at home during the outward-bound part (15 years) of Al's journey; thus, 15 flashes are seen by Bill during the 45 years between Al's departure and his turn-around. Thereafter, the flashes are received every 4 months; thus 15 flashes are seen by Bill in the last 5 years before Al's return. Therefore, Bill sees a total of 30 of Al's birthday fireworks over a period of 50 years of his time.

2.11 Velocity addition in the twin paradox The speed of the rocketship before its turn-around is $\beta_1 = 4/5$ and after the turn-around it is $\beta_2 = -4/5$. Hence, the relative speed between the spaceship, before and after the turn-around, can be computed using the velocity addition rule of (2.22) $\beta_{12} = (\beta_1 - \beta_2)/(1 - \beta_1\beta_2) = 40/41$, which corresponds to a gamma factor of

$\gamma_{12} = (1 - \beta_{12}^2)^{-1/2} = 41/9$. Hence a reading of $t_1 = 9$ years by the clock on the rocketship before the turn-around will be seen by the clock after the turn-around, according to the SR time-dilation, to be $t_2 = \gamma_{12} t_1 = 41$ years.

3.2 **Contraction and dummy indices** After interchanging both pairs of indices, the symmetry properties of these two tensors yield $T_{\mu\nu}S^{\mu\nu} = -T_{\nu\mu}S^{\nu\mu}$. Since we can rename the dummy indices (e.g. rename μ as ν, and ν as μ) we have $T_{\mu\nu}S^{\mu\nu} = -T_{\nu\mu}S^{\nu\mu} = -T_{\mu\nu}S^{\mu\nu}$. Thus $T_{\mu\nu}S^{\mu\nu}$ equals to the negative of itself; it can only be zero.

3.4 **Orthogonality fixes the rotation matrix** The orthogonal condition

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

includes the diagonal conditions of $a^2 + b^2 = c^2 + d^2 = 1$, which can be solved by the parametrization of $a = \cos\phi$, $b = \sin\phi$ and $c = \sin\phi'$, $d = \cos\phi'$; while the off-diagonal condition of $ac + bd = \sin(\phi + \phi') = 0$ implies $\phi = -\phi'$. In terms of the actual rotation angle θ, we make the identification $\phi = \theta$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

3.5 **Group property of Lorentz transformations** We shall only display the group property of the boost transformation: Given the Lorentz boost (3.22), we have the combined transformation

$$[\mathbf{L}(\psi_1)][\mathbf{L}(\psi_2)] = \begin{pmatrix} c_1 & s_1 \\ s_1 & c_1 \end{pmatrix}\begin{pmatrix} c_2 & s_2 \\ s_2 & c_2 \end{pmatrix}$$

where $c_1 \equiv \cosh\psi_1$ and $s_1 \equiv \sinh\psi_1$. A straightforward matrix multiplication and the trigonometric identities, $c_{12} \equiv \cosh(\psi_1 + \psi_2)$ and $s_{12} \equiv \sinh(\psi_1 + \psi_2)$, of $c_{12} = c_1c_2 + s_1s_2$ and $s_{12} = s_1c_2 + c_1s_2$, lead us to

$$[\mathbf{L}(\psi_1)][\mathbf{L}(\psi_2)] = \begin{pmatrix} c_{12} & s_{12} \\ s_{12} & c_{12} \end{pmatrix} = [\mathbf{L}(\psi_1 + \psi_2)],$$

which is the stated result.

3.6 **Group multiplication leads to velocity addition rule** With the identification of (3.25) $\beta = -\tanh\psi$ so that $u/c = \beta_1 = -\tanh\psi_1$ and $-v/c = \beta_2 = -\tanh\psi_2$, and the group multiplication of (3.58), $u'/c = \beta_{12} = -\tanh\psi_{12} = -\tanh(\psi_1 + \psi_2)$, the velocity addition rule (2.22) follows from the trigonometric identity of

$$\tanh(\psi_1 \pm \psi_2) = \frac{\tanh\psi_1 \pm \tanh\psi_2}{1 \pm \tanh\psi_1 \tanh\psi_2}.$$

3.7 **Lorentz transform and velocity addition rule** Suppressing the transverse spatial components, the 4-velocities, according to (3.31), have components (in self-evident notations) $U^\mu = \gamma_u(c, u)$ and $U'^\mu = \gamma'_u(c, u')$, which are connected by Lorentz transformation $U'^\mu = \mathbf{L}^\mu_\nu U^\nu$

$$\begin{pmatrix} \gamma'_u c \\ \gamma'_u u' \end{pmatrix} = \begin{pmatrix} \gamma_v & -\gamma_v\beta_v \\ -\gamma_v\beta_v & \gamma_v \end{pmatrix}\begin{pmatrix} \gamma_u c \\ \gamma_u u \end{pmatrix} = c\gamma_v\gamma_u \begin{pmatrix} 1 - \beta_v\beta_u \\ -\beta_v + \beta_u \end{pmatrix}.$$

Equating the first-row elements leads to $\gamma'_u = \gamma_v \gamma_u (1 - \beta_v \beta_u)$. When this is substituted into the equality of the second-row elements, we obtain the velocity addition rule of (2.22).

3.8 **Antiproton production threshold** The minimum energy needed to produce the final state of three protons and one antiproton in the center-of-mass frame is $E_{\text{final}} = 4\,mc^2$. The square of the total 4-momentum of the final state, given the total 3-momentum being zero (because of CM frame), $p^\mu_{\text{final}} = (c^{-1}E_{\text{final}}, \vec{0})$ must then be $\eta_{\mu\nu} p^\mu_{\text{final}} p^\nu_{\text{final}} = -16\,m^2c^2$. By energy–momentum conservation, this must also the square of the total 4-momentum of the initial state of two protons (projectile and target): $p^\mu_{\text{final}} = p^\mu_{\text{initial}}$, where p^μ_{initial} is the initial total 4-momentum with total energy and total 3-momentum of the initial state, denoted by $(E, \vec{p})$, as its components. We then have, from (3.38),

$$-16m^2c^4 = -E^2 + |\vec{p}|^2\,c^2. \tag{5}$$

In the lab frame, in which the target proton is at rest, we have $E = E_1 + mc^2$, where E_1 is the energy of the projectile proton. The 3-momentum is given entirely by the projectile proton $\vec{p} = \vec{p}_1$, which is related to E_1 by the usual energy–momentum relation: $|\vec{p}|^2c^2 = |\vec{p}_1|^2c^2 = E_1^2 - m^2c^4$. Substitute these two relations into (5) and solve for the projectile proton's lab energy to get $E_1 = 7\,mc^2$, which corresponds to a kinetic energy of the projectile $K_{\text{lab}} = E_1 - mc^2 = 6\,mc^2 = 5.6$ GeV.

3.9 **More conventional derivation of Doppler effect** Since the sender is at rest, we have $t = \tau$ and $d\phi = \omega dt$; on the other hand, for the moving receiver, we have $d\phi = \omega' dt'/\gamma = \sqrt{1-\beta^2}\omega' dt'$. Thus invariance of the phase leads to $(\omega'/\omega) = (dt/dt')/\sqrt{1-\beta^2}$. Now the two events (x, t) and (x', t') are connected by a light signal, we have $(x' - x) = c(t' - t)$ or $(dt/dt') = 1 - \beta$ (after using $dx'/dt' = v$). In this way we obtain

$$\frac{\omega'}{\omega} = \frac{1-\beta}{\sqrt{1-\beta^2}} = \sqrt{\frac{1-\beta}{1+\beta}}.$$

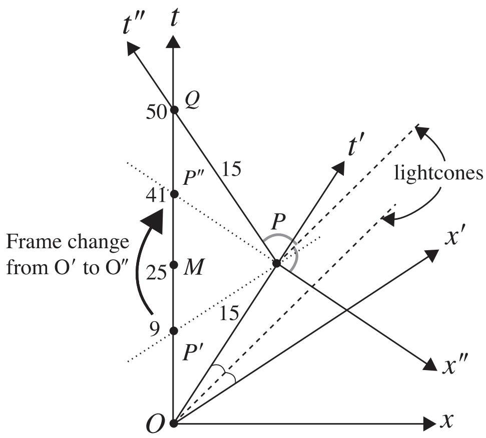
3.10 **Twin paradox measurements and Doppler effect** We can view the sending and observing birthday fireworks as the sending and receiving of light signals. Thus the respective emission and receiving frequencies should obey the Doppler relation (3.47). The relative velocities for the outward and inward bound trips being $\beta = \pm 4/5$, the formula yields $\omega' = \omega/3$, and $\omega' = 3\omega$, respectively. This is just the frequency changes of birthday fireworks observed, for example, during the outward bound part $\beta = 4/5$, the emission frequency is red-shifted to $\omega' = \omega/3$ and Al sees Bill's annual firework every three years.

3.11 **Spacetime diagram for the twin paradox** **(see displayed diagram.)**

3.12 **The twin paradox – the missing 32 years**

(a) The gamma factor between O and O'' frames being $\gamma = 5/3$, Al's inbound 15 years corresponds to the last 9 years ($= t_{QP''}$) of the total $t_Q = 50$ years. Hence $t_{P''} = t_Q - t_{QP''} = 41$ years.

(b) First we need to calculate the time dilation factor between O' and O'' frames. For this we need to work out the relative velocity $\bar{\beta}$ of these two frames. We can deduce it from the relative velocities $\beta_{1,2} = \pm 4/5$ of these two frames with respect to the O system by using the velocity addition rule of (2.22), $\bar{\beta} = (\beta_1 - \beta_2)/(1 - \beta_1\beta_2) = 40/41$. This gives rise to a



Problem 3.11 Three worldlines of the twin paradox: *OQ* is that for the stay-at-home Bill, *OP* that for the outward-bound part ($\beta = 4/5$) , *PQ* that for the inward-bound part ($\beta = -4/5$) of the Al's journey. *M* is the midpoint between *O* and *Q*. These three lines define three inertial frames: O, O', and O'' systems. When Al changes from the O' to the O'' system at *P* the point that's simultaneous (with *P*) along Bill's worldline OQ jumps from point P' to P''. From the viewpoint of Bill, this is a leap of 32 years.

$\bar{\gamma} = 41/9$. We then find the turning point P as having an O'' frame time of $t''_{OP} = \bar{\gamma} t'_P = (41/9) \times 15 = (205/3)$ years. This moving clock time corresponds to the rest-frame time of $t_{P''} = (205/3)/(5/3) = 41$ years.

3.13 **Spacetime diagram for the pole-and-barn paradox (see displayed diagram.)**

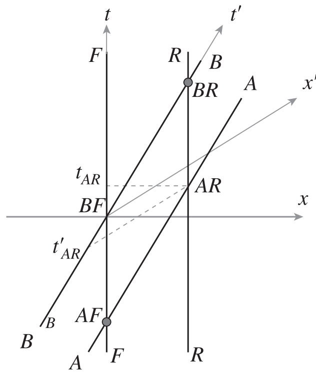
4.1 **Inclined plane, pendulum and EP**

(a) **Inclined plane** The $F = ma$ equation along the inclined plane, is $m_\mathrm{I} a = m_\mathrm{G} g \sin\theta$, leading to a material-dependent acceleration: $a_\mathrm{A} = g \sin\theta \left(\frac{m_\mathrm{G}}{m_\mathrm{I}}\right)_\mathrm{A}$.

(b) **Pendulum** For the simple pendulum with a light string of length L, we have $m_\mathrm{I} L (d^2\theta/dt^2) = -m_\mathrm{G} g \sin\theta$. This has the form of a simple harmonic oscillator equation when approximated by $\sin\theta \approx \theta$, leading to a period of

$$T_\mathrm{A} = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{L}{g}\left(\frac{m_\mathrm{I}}{m_\mathrm{G}}\right)_\mathrm{A}}$$

for a blob made up of material A.



Problem 3.13 Spacetime diagram for the pole-and-barn paradox. Ground (barn) observer has (x, t) coordinates, while the runner (pole) rest frame has (x', t') coordinates. The heavy lines are the worldlines for the front-door (F), rear-door (R) of the barn, and front-end (A), back-end (B) of the pole. Note the order reversal: $t_{AR} > t_{BF}$ and $t'_{AR} < t'_{BF}$.

4.2 **Two EP brain-teasers**

(a) **Forward leaning balloon** According to EP the effective gravity is the vector sum $\mathbf{g}_\mathrm{eff} = \mathbf{g} + (-\mathbf{a})$, where $\mathbf{g}$ is the normal gravity (pointing vertically downward) while $\mathbf{a}$ is the acceleration of the vehicle. The buoyant force is always opposite to $\mathbf{g}_\mathrm{eff}$.

(b) **A toy for Einstein** Normally what is difficult to do is to have a net force pulling the ball back into the bowl. The net force is the combination of gravity and spring restoring force. But the task can be made easy by dropping the whole contraption—because EP informs us that gravity would disappear in this freely falling system. Without the interference of gravity, the spring will pull back the ball each time without any difficulty.

4.3 **The Global Position System**

(a) A satellite's centripetal acceleration is produced by earth's gravity: $v_s^2/r_s = G_\mathrm{N} M_\oplus / r_s$. The orbit period T_s is related to the radius and tangential velocity: $T_s = 2\pi r_s / v_s$. Knowing that $T_s = 12$ h $= 4.32 \times 10^4$ s we can find r_s and v_s from these two equations: $r_s \simeq 2.7 \times 10^7$ m $\simeq 4.2R_\oplus$, $v_s \simeq$ 3.9 km/s, where $R_\oplus$ is earth's radius.

(b) The SR time dilation factor being $\gamma_s = (1-\beta_s^2)^{-1/2} = 1 + \beta_s^2/2 + \cdots$ the fractional change is then $t/\tau - 1 = \gamma_s - 1 = \beta_s^2/2 = (v_s/c)^2/2 \simeq 0.85 \times 10^{-10}$. Here we have neglected the rotational speed of the clock on the ground—the corresponding β^2 value is a hundred times smaller even for the largest value on the equator.

(c) The gravitational time dilation effect is given by (4.31) with $\Phi = -G_{\rm N}M/r$:

$$\frac{\Phi_\oplus - \Phi_s}{c^2} = -G_{\rm N}\frac{M_\oplus}{c^2}\left(\frac{1}{R_\oplus} - \frac{1}{r_s}\right) \simeq -5.2 \times 10^{-10}.$$

Thus, the general relativity (GR) effect is about six times larger than the special relativity (SR) effect.

(d) In one minute duration $(\Delta t)_{\rm GR} \simeq -30$ ns, and $(\Delta t)_{\rm SR} \simeq 5$ ns. The gravitational effect makes the satellite clock go faster because it is at a higher gravitational potential. The SR dilation slows it down. The net effect is to make the clock in the satellite, when compared to the clock on the ground, run faster by about 25 ns for every passage of 1 min. This translates into a distance of $(2.5 \times 10^{-8}$ s$) \times (3 \times 10^8$ m/s$) = 7.5$ m.

Here is an example of the practical application in our daily life of this "pure science" of general relativity.

4.4 **Gravitational redshift directly from Doppler effect** The receiver being in motion, moving with nonrelativistic velocity $\beta = \Delta u/c$, the SR Lorentz frequency transformation (3.47) becomes

$$\frac{\omega_{\rm rec}}{\omega_{\rm em}} = \sqrt{\frac{1-\beta}{1+\beta}} \simeq 1 - \beta.$$

Or, equivalently $\Delta\omega/\omega = -\Delta u/c$, which is just the expression shown in (4.22) leading to the final gravitational result of (4.24).

5.2 **Basis vectors on a spherical surface** The respective basis vectors are

$$\mathbf{e}_\theta = R\hat{\mathbf{u}}_\theta, \quad \mathbf{e}_\phi = R\sin\theta\hat{\mathbf{u}}_\phi,$$

where $\hat{\mathbf{u}}_\theta$ is the unit vector in the polar angle direction, and $\hat{\mathbf{u}}_\phi$ is perpendicular to $\hat{\mathbf{u}}_\theta$ in the azimuthal direction. The resultant metric matrix, according to Eq. (5.7), is

$$g_{ab} = \begin{pmatrix} \mathbf{e}_\theta\cdot\mathbf{e}_\theta & \mathbf{e}_\theta\cdot\mathbf{e}_\phi \\ \mathbf{e}_\phi\cdot\mathbf{e}_\theta & \mathbf{e}_\phi\cdot\mathbf{e}_\phi \end{pmatrix} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix}.$$

5.3 **Coordinate transformation of the metric** Given the transformation (5.18), we have the inverse matrix

$$\mathbf{R}^{-1} = \begin{pmatrix} (R\cos\theta)^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

Using the metric transformation condition given in sidenote 11 of Chapter 3, we can obtain the metric for the cylindrical system from that of the polar system

$$\begin{aligned}\mathbf{R}^{-1\intercal}\mathbf{g}\mathbf{R}^{-1} &= \mathbf{R}^{-1\intercal}\begin{pmatrix}R^2 & 0\\ 0 & R^2\sin^2\theta\end{pmatrix}\mathbf{R}^{-1}\\ &= \begin{pmatrix}(\cos^2\theta)^{-1} & 0\\ 0 & R^2\sin^2\theta\end{pmatrix} = \begin{pmatrix}1-(\rho^2/R^2)^{-1} & 0\\ 0 & \rho^2\end{pmatrix}\\ &= \mathbf{g}'.\end{aligned}$$

5.4 **Geodesics on simple surfaces**

(a) **Flat plane** For this 2D space with Cartesian coordinates $(x^1, x^2) = (x, y)$, the metric $g_{ab} = \delta_{ab}$. The second term in the geodesic Eq. (5.30) vanishes, as well as the two components of the equation $d\dot{x}^\nu/d\lambda$ so that $\ddot{x} = 0$ and $\ddot{y} = 0$, which have respective solutions of $x = \mathrm{A} + \mathrm{B}\lambda$ and $y = \mathrm{C} + \mathrm{D}\lambda$. They can be combined as $y = \alpha + \beta x$, with (A, B, C, D) and (α, β) being constants. We recognize this as the equation for a straight line.

(b) **Spherical surface** For a 2-sphere, we choose the coordinates $(x^1, x^2) = (\theta, \phi)$ with a metric given by (5.13) For the θ component of the geodesic Eq. (5.30) is $\ddot{\theta} = \sin\theta\cos\theta\dot{\phi}^2$, the ϕ component equation, $2\sin\theta\cos\theta\dot{\theta}\dot{\phi} + \sin^2\theta\ddot{\phi} = 0$. Instead of working out the full parametric representation, we will just check that $\phi =$ constant and $\theta = \alpha + \beta\lambda$ solve these two equations. Clearly these solutions describe longitudinal great circles on the sphere.

5.5 **Locally flat metric** The distance between two neighboring points can be rearranged by adding and subtracting a factor of $(g_{12}dx^2)^2/g_{11}$ so that

$$\begin{aligned}ds^2 &= g_{11}(dx^1)^2 + 2g_{12}dx^1dx^2 + g_{22}(dx^2)^2\\ &= \left(\sqrt{g_{11}}dx^1 + \frac{g_{12}dx^2}{\sqrt{g_{11}}}\right)^2 + \left(g_{22} - \frac{g_{12}^2}{g_{11}}\right)(dx^2)^2.\end{aligned}$$

The new coordinate system $(\bar{x}^1, \bar{x}^2)$ has the metric $\bar{g}_{ab} = \delta_{ab}$ because $ds^2 = (d\bar{x}^1)^2 + (d\bar{x}^2)^2$ where

$$d\bar{x}^1 = \sqrt{g_{11}}dx^1 + \frac{g_{12}dx^2}{\sqrt{g_{11}}}, \quad d\bar{x}^2 = \sqrt{g_{22} - \frac{g_{12}^2}{g_{11}}}dx^2.$$

On the other hand, if the original metric determinant is negative, $g_{11}g_{22} - g_{12}^2 < 0$, then $ds^2 = (d\bar{x}^1)^2 - (d\bar{x}^2)^2$ with

$$d\bar{x}^2 = \sqrt{\frac{g_{12}^2}{g_{11}} - g_{22}}dx^2.$$

5.7 **3-sphere and 3-pseudosphere**

(a) **3D flat space**

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta.$$

The relation for the solid angle factor follows simply from the two expressions for the invariant separations in two coordinate systems:

$$ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\Omega^2.$$

(b) **3-sphere** Given the metric for 3-sphere being

$$ds^2 = dr^2 + \left(R \sin\frac{r}{R}\right)^2 d\Omega^2, \tag{6}$$

the relation from part (a) $r^2 d\Omega^2 = dx^2 + dy^2 + dz^2 - dr^2$ suggests

$$\left(R \sin\frac{r}{R}\right)^2 d\Omega^2 = dX^2 + dY^2 + dZ^2 - \left[d\left(R \sin\frac{r}{R}\right)\right]^2.$$

Substituting this into (6), we have

$$ds^2 = dW^2 + dX^2 + dY^2 + dZ^2$$

where

$$dW^2 = dr^2 - \left[d\left(R \sin\frac{r}{R}\right)\right]^2 = \left[\sin\frac{r}{R} dr\right]^2$$

so we can identify $dW = \sin(r/R)dr$. This ds^2 invariant interval implies a Euclidian metric $g_{\mu\nu} = \mathrm{diag}(1, 1, 1, 1)$. Also, it suggests the embedding relation between (r, θ, ϕ) and (W, X, Y, Z) as

$$W = R\cos\frac{r}{R}, \quad X = \left(R\sin\frac{r}{R}\right)\sin\theta\cos\phi,$$
$$Y = \left(R\sin\frac{r}{R}\right)\sin\theta\sin\phi, \quad Z = \left(R\sin\frac{r}{R}\right)\cos\theta.$$

This set of relations lead immediately to the constraint $W^2 + X^2 + Y^2 + Z^2 = R^2$.

(c) **3-pseudosphere** With $W = R\cosh(r/R)$, the relations

$$W = R\cosh\frac{r}{R}, \quad X = \left(R\sinh\frac{r}{R}\right)\sin\theta\cos\phi,$$
$$Y = \left(R\sinh\frac{r}{R}\right)\sin\theta\sin\phi, \quad Z = \left(R\sinh\frac{r}{R}\right)\cos\theta,$$

lead, through the trigonometric relation $\cosh^2\chi - \sinh^2\chi = 1$ to

$$ds^2 = -dW^2 + dX^2 + dY^2 + dZ^2$$

thus a Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ and the condition

$$-W^2 + X^2 + Y^2 + Z^2 = -R^2.$$

5.8 Volume of higher dimensional space

$$dV = \sqrt{\det g}\prod_i dx^i. \tag{7}$$

(a) **3D flat space** For Cartesian coordinates $\sqrt{\det g} = 1$, (7) reduces to $dV = dxdydz$, and for spherical coordinates $\sqrt{\det g} = r^2 \sin\theta$ and $dV = r^2 \sin\theta\, dr d\theta\, d\phi$.

(b) **3-sphere** From (7) we have $\sqrt{\det g} = R^2 \sin^2(r/R) \sin\theta$, thus the volume of a 3-sphere with radius R can be calculated:

$$R^2 \int_0^{\pi R} \sin^2 \frac{r}{R} dr \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi = 2\pi^2 R^3.$$

5.9 **Non-Euclidean relation between radius and circumference of a circle**

(a) **The case of a sphere** The radius of a circle being the displacement ds along a constant radial coordinate ($dr = 0$), we have from either (5.48), (5.49) or (5.51), (5.52), $ds = R\sin(r/R)d\phi$. Thus, making a Taylor series expansion of the circumference $S = \int ds$, we have:

$$\begin{aligned} S = 2\pi R \sin \frac{r}{R} &= 2\pi R \left(\frac{r}{R} - \frac{1}{3}\frac{r^3}{R^3} + \cdots \right) \\ &= 2\pi r - \frac{1}{R^2}\frac{\pi r^3}{3} + \cdots \end{aligned}$$

which is just the claimed result in (5.39) with $K = 1/R^2$.

(b) **The case of a pseudosphere**: For $k = -1$ surface, the displacement, according to either (5.48), (5.49) or (5.51), (5.52), is given by $ds = R\sinh(r/R)d\phi$, giving a circumference of $S = 2\pi R \sinh(r/R)$. Since the Taylor expansion of the hyperbolic sine differs from that for the sine function in the sign of the cubic term, again we obtain the result in agreement with (5.39) with $K = -1/R^2$. Thus, on a pseudospherical surface, the circumference of a circle with radius r is $S > 2\pi r$.

5.10 **Angular excess and polygon area** Any polygon is made up of triangles.

5.11 **Local Euclidean coordinates** From the chain rule of differentiation,

$$\begin{aligned} d\xi^1 &= \frac{\partial \xi^1}{\partial x^1} dx^1 + \frac{\partial \xi^1}{\partial x^2} dx^2 \\ d\xi^2 &= \frac{\partial \xi^2}{\partial x^1} dx^1 + \frac{\partial \xi^2}{\partial x^2} dx^2. \end{aligned}$$

Equate $ds^2 = (d\xi^1)^2 + (d\xi^2)^2 = g_{11}(dx^1)^2 + 2g_{12}(dx^1)(dx^2) + g_{22}(dx^2)^2$ we can immediately get the relation we are looking for:

$$\begin{aligned} g_{11} &= \left(\frac{\partial \xi^1}{\partial x^1}\right)^2 + \left(\frac{\partial \xi^2}{\partial x^1}\right)^2 \\ g_{12} &= \left(\frac{\partial \xi^1}{\partial x^1}\right)\left(\frac{\partial \xi^1}{\partial x^2}\right) + \left(\frac{\partial \xi^2}{\partial x^1}\right)\left(\frac{\partial \xi^2}{\partial x^2}\right) \\ g_{22} &= \left(\frac{\partial \xi^1}{\partial x^2}\right)^2 + \left(\frac{\partial \xi^2}{\partial x^2}\right)^2. \end{aligned}$$

For the spherical polar coordinates $(x^1, x^2) = (\theta, \phi)$, we have $\xi^1 = R\theta$ and $\xi^2 = (R\sin\theta)\phi$ with their direction be along (θ, ϕ) so that $d\xi^1 = Rd\theta$ and $d\xi^2 = (R\sin\theta)d\phi$. In this way, we have

$$\left(\frac{\partial \xi^1}{\partial x^2}\right) = \left(\frac{\partial \xi^2}{\partial x^1}\right) = 0$$

to obtain the metric element $g_{12} = 0$ and

$$g_{11} = \left(\frac{\partial \xi^1}{\partial x^1}\right)^2 = R^2, \quad g_{22} = \left(\frac{\partial \xi^2}{\partial x^2}\right)^2 = R^2 \sin^2\theta.$$

6.2 **Spatial distance and spacetime metric**
The spacetime separation vanishes ($ds^2 = 0$) for a light pulse:

$$g_{00}\left(dx^0\right)^2 + 2g_{0i}dx^i dx^0 + g_{ij}dx^i dx^j = 0.$$

Solving this quadratic equation for the coordinate time interval that takes the pulse going from A to B

$$dx^0_{\text{AB}} = -\frac{g_{0i}dx^i}{g_{00}} - \frac{\sqrt{(g_{0i}g_{0j} - g_{00}g_{ij})dx^i dx^j}}{g_{00}}$$

and the time for it to go from B to A (involving the change of $dx^i \to -dx^i$)

$$dx^0_{\text{BA}} = +\frac{g_{0i}dx^i}{g_{00}} - \frac{\sqrt{(g_{0i}g_{0j} - g_{00}g_{ij})dx^i dx^j}}{g_{00}}.$$

Therefore the total coordinate time

$$dx^0 = dx^0_{\text{AB}} + dx^0_{\text{BA}},$$

which is related to the proper time interval $d\tau_{\text{A}}$ (see Problem 6.1), hence the spatial distance dl,

$$dl \equiv \frac{cd\tau_{\text{A}}}{2} = \frac{\sqrt{-g_{00}}dx^0}{2} = \sqrt{\left(g_{ij} - \frac{g_{0i}g_{0j}}{g_{00}}\right)dx^i dx^j}.$$

Since $dl^2 = \gamma_{ij}dx^i dx^j$, we have $\gamma_{ij} = g_{ij} - (g_{0i}g_{0j}g_{00})$. Thus, $\gamma_{ij} \neq g_{ij}$ when $g_{0i} \neq 0$.

6.3 **Non-Euclidean geometry of a rotating cylinder** Let us denote the spatial coordinates as follows:

(ct, r, ϕ, z) lab observer,

(ct, r_0, ϕ_0, z) observer on the rotating disk.

They are related by (see Fig. 6.1)

$$r = r_0, \quad \phi = \phi_0 + \omega t.$$

We shall ignore the vertical coordinate z below.

The line element written in terms coordinates at rest with respect to the observer on the rotating disk is, see (5.33)

$$ds^2 = -c^2 dt^2 + dr_0^2 + r_0^2 d\phi_0^2,$$

which can be written in terms of the lab coordinate (see Cook, 2004) by substituting in $d\phi_0 = d\phi - \omega dt$:

$$ds^2 = -\left[1 - \left(\frac{\omega r}{c}\right)^2\right] c^2 dt^2 + dr^2 + r^2 d\phi^2 - 2\omega r^2 dt d\phi.$$

The metric with respect to the (ct, r, ϕ) coordinates thus has elements

$$g_{00} = -\left[1 - \left(\frac{\omega r}{c}\right)^2\right], \quad g_{rr} = 1, \quad g_{\phi\phi} = r^2, \quad g_{0\phi} = -\frac{\omega r^2}{c}.$$

From Problem 6.2, we have the spatial distance

$$dl^2 = \left(g_{ij} - \frac{g_{0i} g_{0j}}{g_{00}}\right) dx^i dx^j = dr^2 + \frac{r^2 d\phi^2}{1 - (\omega r/c)^2}$$

showing clearly length contraction of the circumference, but not the radius.

6.5 **The geodesic equation and light deflection** The geodesic equation (6.9), after using $p^\mu = dx^\mu / d\lambda$, can be written as

$$\frac{d}{d\lambda} p^\mu + \Gamma^\mu_{\nu\sigma} p^\nu \frac{dx^\sigma}{d\lambda} = 0$$

or equivalently

$$dp^\mu = -\Gamma^\mu_{\nu\sigma} p^\nu dx^\sigma.$$

We are interested in the $\mu = 2$ component $p^2 \equiv p_y$:

$$\begin{aligned} dp_y &= -\Gamma^2_{00} p^0 dx^0 - \Gamma^2_{11} p^1 dx^1 - \Gamma^2_{10} p^1 dx^0 - \Gamma^2_{01} p^0 dx^1 \\ &= -(\Gamma^2_{00} + \Gamma^2_{11} + 2\Gamma^2_{10}) p dx, \end{aligned} \tag{8}$$

where we have used $dx^\mu = (dx, dx, 0, 0)$ and $p^\mu = (p, p, 0, 0)$. Christoffel symbols can be calculated by (6.10). Since we are working in the weak-field approximation, that is, the metric is very close to being the flat space Minkowski metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$, and the Christoffel symbols (being the derivatives of the metric) must also be small. Thus the metric on the left-hand side (LHS) of (6.10) can be taken to be $\eta_{\mu\nu}$, which is diagonal. Consider the LHS component $\eta_{2\sigma} \Gamma^\sigma_{\mu\nu} = \Gamma^\sigma_{\mu\nu}$:

$$\Gamma^2_{\mu\nu} = \frac{1}{2}\left[\frac{\partial g_{\mu 2}}{\partial x^\nu} + \frac{\partial g_{\nu 2}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^2}\right].$$

If the only position-dependent metric element is $g_{00} = -1 - \Phi/c^2$ (as suggested by EP physics), then the only nonzero term on the RHS is the $\partial g_{00}/\partial x^2$.

That is,

$$\Gamma^2_{00} = \frac{-1}{c^2}\frac{\partial\Phi}{\partial y}$$

and Eq. (8) reduces to $dp_y = -\Gamma^2_{00} p dx$. This way we get

$$\delta\phi_{\text{EP}} = \int \frac{dp_y}{p} = \frac{1}{c^2}\int \frac{\partial\Phi}{\partial y} dx \tag{9}$$

which is the result obtained by Huygens' principle in Eq. (4.44). For the argument that the GR value is twice that of the EP value, see Sections 7.2.1 and 7.3.2.

6.7 **The matrix for tidal forces is traceless** We can take the trace of the tidal force matrix by contracting the two indices with the Kronecker delta:

$$\delta_{ij}\frac{\partial^2\Phi}{\partial x^i \partial x^j} = \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^i}\Phi = \nabla^2\Phi.$$

Since the mass density vanishes ($\rho = 0$) at any field point away from the source, the Newtonian field equation (6.5) informs us that the gravitational potential satisfies the Laplace equation $\nabla^2\Phi = 0$.

6.8 **G_{N} as a conversion factor** One easily finds that this yields the dimension relation (curvature) = (length)$^{-2}$. This is consistent with the fact that curvature is the second derivative of the metric, which is dimensionless.

7.1 **Energy relation for a particle moving in the Schwarzschild spacetime** Equation (7.44) with $r^* = 0$ is

$$-c^2\left(\frac{dt}{d\tau}\right)^2 + \left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2 = -c^2,$$

where τ is the proper time $d/d\tau = \gamma d/dt$ with $\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$. Multiplying a factor of $-m^2c^2$ on both sides, we obtain

$$\gamma^2 m^2\left[c^4 - c^2v^2\right] = m^2c^4,$$

where $v^2 = (dr/dt)^2 + r^2(d\phi/dt)^2$ is the velocity (squared) in the spherical coordinate system (r, θ, ϕ) when the polar angle θ is fixed. We recognize this is the energy–momentum relation $E^2 = p^2c^2 + m^2c^4$ after identifying the relativistic expression for energy $E = \gamma mc^2$ and momentum $p = \gamma mv$.

7.2 **Gravitational red shift via energy conservation** The frequency ratio being

$$\frac{\omega_{\text{em}}}{\omega_{\text{rec}}} = \frac{\left(g_{00}p^0\right)_{\text{em}} U^0_{\text{em}}}{\left(g_{00}p^0\right)_{\text{rec}} U^0_{\text{rec}}} = \frac{U^0_{\text{em}}}{U^0_{\text{rec}}},$$

where to reach the last equality we have used the energy conservation relation of $g_{\mu\nu}p^\mu K^\nu_{(t)\text{em}} = g_{\mu\nu}p^\mu K^\nu_{(t)\text{rec}}$ with the Killing vector $K^\mu_{(t)} = (1, 0, 0, 0)$. The invariance of the 4-velocity squared $(g_{00}U^0U^0)_{\text{em}} = (g_{00}U^0U^0)_{\text{rec}}$ leads to the desired result of

$$\frac{\omega_{\text{em}}}{\omega_{\text{rec}}} = \frac{U^0_{\text{em}}}{U^0_{\text{rec}}} = \sqrt{\frac{(g_{00})_{\text{rec}}}{(g_{00})_{\text{em}}}},$$

which can be translated into the standard expression of $(\Delta\omega)/\omega = -(\Delta\Phi)/c^2$ as done in Section 6.2.2.

7.3 **Light deflection via the geodesic equation** The $L = ds^2/d\lambda^2 = 0$ equation in the form of (7.77) can be written slightly differently as

$$\left(\frac{dr}{d\lambda}\right)^2 + \left(1 - \frac{r^*}{r}\right)\frac{\lambda^2}{r^2} = \kappa^2.$$

After the usual change of variables $(r, \lambda) \to (u, \phi)$, we have

$$u'' + u - \epsilon u^2 = 0.$$

For a perturbative solution of $u = u_0 + \epsilon u_1$ with $\epsilon = O\left(r^*\right)$,

$$\left(u_0'' + u_0\right) + \epsilon\left(u_1'' + u_1 - u_0^2\right) + \cdots = 0.$$

The zeroth order, being a "simple harmonic oscillator" equation, has the solution $u_0 = r_{\text{min}}^{-1}\sin\phi$. To solve the first order equation

$$\frac{d^2u_1}{d\phi^2} + u_1 = \frac{1 - \cos 2\phi}{2r^2_{\text{min}}}$$

one tries $u_1 = \alpha + \beta\cos 2\phi$, and finds $\alpha = (2r^2_{\text{min}})^{-1}$ and $\beta = (6r^2_{\text{min}})^{-1}$. Putting the zeroth and first order terms together we get

$$\frac{1}{r} = \frac{\sin\phi}{r_{\text{min}}} + \frac{3 + \cos 2\phi}{4}\frac{r^*}{r^2_{\text{min}}}.$$

In the absence of gravity ($r^* = 0$), the asymptotes ($r = \mp\infty$) corresponds to $\phi_{-\infty} = \pi$ and $\phi_{+\infty} = 0$, and the trajectory is straight line (no deflection). When gravity is turned on, there is an angular deflection $\delta\phi = (\phi_{-\infty} - \phi_{+\infty} - \pi)$. Picking our coordinates so that $\phi_{-\infty} = \pi + \delta\phi/2$ and $\phi_{+\infty} = -\delta\phi/2$ and the trajectory equations yields (for either asymptote):

$$0 = -\sin\frac{\delta\phi}{2} + \frac{3 + \cos\delta\phi}{4}\frac{r^*}{r_{\text{min}}}.$$

For small deflection angle $\delta\phi$ we have $0 = -\delta\phi/2 + r^*/r_{\text{min}}$; we obtain the result of $\delta\phi_{\text{GR}} = 2r^*/r_{\text{min}}$.

7.5 **Total energy in curved spacetime** We can check this claim by showing that starting with $\kappa/c = E/mc^2$ one can deduce $\mathcal{E} = E - mc^2$ in the NR limit. From the definition, we have

$$\mathcal{E} \equiv m\left(\kappa^2 - c^2\right)/2 = \frac{1}{2}mc^2\left[\left(\frac{E}{mc^2}\right)^2 - 1\right].$$

The NR total energy $e_{\rm NR}$ is defined in the NR limit by $E = mc^2 + e_{\rm NR}$ with $mc^2 \gg e_{\rm NR}$. Thus, the above equation does turn into $\mathcal{E} = e_{NR}$ in the NR limit.

7.6 **Details for time-delay calculation**

We substitute in the expansion of b given in (7.80) and expand

$$\begin{aligned}
c\frac{dt}{dr} &= \left(1 - \frac{r^*}{r}\right)^{-1}\left[1 - \frac{b^2}{r^2}\left(1 - \frac{r^*}{r}\right)\right]^{-1/2} \\
&\simeq \left(1 + \frac{r^*}{r}\right)\left[1 - \left(\frac{r_0^2}{r^2} + \frac{r_0 r^*}{r^2}\right)\left(1 - \frac{r^*}{r}\right)\right]^{-1/2} \\
&\simeq \left(1 + \frac{r^*}{r}\right)\left[1 - \frac{r_0^2}{r^2} - \frac{r_0 r^*}{r^2} + \frac{r_0^2 r^*}{r^3}\right]^{-1/2} \\
&\simeq \left(1 + \frac{r^*}{r}\right)\left(1 - \frac{r_0^2}{r^2}\right)^{-1/2}\left[1 + \frac{1}{2}\frac{\frac{r_0 r^*}{r^2}}{1 + \frac{r_0}{r}}\right] \\
&\simeq \left(1 - \frac{r_0^2}{r^2}\right)^{-1/2}\left[1 + \frac{r^*}{r}\left(1 + \frac{\frac{1}{2}r_0}{r + r_0}\right)\right] \\
&= \left(1 - \frac{r_0^2}{r^2}\right)^{-1/2}\left[1 + \frac{r^*}{r}\frac{r + \frac{3}{2}r_0}{r + r_0}\right].
\end{aligned}$$

7.7 **4-velocity of a particle in a circular orbit**

(a) The orbit equation (7.59) for the variable $u \equiv 1/r$ has $u' = 0$ corresponding to a circular orbit case: $u^2 - (r^* c^2/\lambda^2)u - r^* u^3 =$ constant, with $\lambda = l/m = r^2 d\phi/d\tau$. If we differentiate this equation, we have $(r^* c^2/\lambda^2) = 2u - 3r^* u^2$. Putting in the specific value $u = 1/R$, it implies

$$R^4\left(\frac{d\phi}{d\tau}\right)^2 = \lambda^2 = \frac{r^* c^2 R}{2}\left(1 - \frac{3r^*}{2R}\right)^{-1}$$

or

$$\left(U^\phi\right)^2 = \left(\frac{d\phi}{d\tau}\right)^2 = \frac{r^* c^2}{2R^3}\left(1 - \frac{3r^*}{2R}\right)^{-1}. \qquad (10)$$

As worked out in Problem (8.5), we can deduce from the energy balance equation (7.52) the conserved time-component of the 4-velocity

$$\left(U^t\right)^2 = \left(\frac{dt}{d\tau}\right)^2 = \left(1 - \frac{3r^*}{2R}\right)^{-1}. \qquad (11)$$

(b) From these two equations (10) and (11), we can easily work out the orbital frequency

$$\begin{aligned}
\Omega^2 \equiv \left(\frac{d\phi}{dt}\right)^2 &= \left(\frac{d\phi}{d\tau}\right)^2\left(\frac{d\tau}{dt}\right)^2 \\
&= \frac{r^* c^2}{2R^3} = \frac{G_{\rm N} M}{R^3},
\end{aligned}$$

which is Kepler's third law. We also note trivially that, for this simple kinematics, the proportionality constant for U^ϕ and U^t is just this orbital frequency: $U^\phi = \Omega U^t$.

(c) Finally we perform the consistency check for the invariant length of the 4-vector. For the particle in the circular orbit ($r = R$) in the equatorial plane ($\theta = \pi/2$), the other two components of the velocity vanihse $U^r = U^\theta = 0$, and at the trajectory the metric elements have values $g_{00} = -(1 - \frac{r^*}{R})$ and $g_{\phi\phi} = R^2$. We see that the 4-velocity indeed has the correct invariant length:

$$\begin{aligned} g_{\alpha\beta} U^\alpha U^\beta &= g_{00}\left(U^0\right)^2 + g_{\phi\phi}\left(U^\phi\right)^2 \\ &= c^2\left(-1 + \frac{r^*}{R} + \frac{r^*}{2R}\right)\left(1 - \frac{3r^*}{2R}\right)^{-1} = -c^2. \end{aligned}$$

8.2 **The coordinate time across an event horizon** In the region outside the Schwarzschild surface, in order to find the full expression of the Schwarzschild coordinate time as a function of the radial distance, one can integrate (from r_0 to r) the equation in (8.5) to obtain

$$\begin{aligned} t = t_0 &- \frac{2r^*}{3c}\left[\left(\frac{r}{r^*}\right)^{\frac{3}{2}} - \left(\frac{r_0}{r^*}\right)^{\frac{3}{2}}\right] \\ &+ \frac{r^*}{c}\left\{\ln\left|\frac{\sqrt{r/r^*}+1}{\sqrt{r/r^*}-1}\cdot\frac{\sqrt{r_0/r^*}-1}{\sqrt{r_0/r^*}+1}\right| - 2\left[\left(\frac{r}{r^*}\right)^{\frac{1}{2}} - \left(\frac{r_0}{r^*}\right)^{\frac{1}{2}}\right]\right\}. \end{aligned} \tag{12}$$

In the limit of r and r_0 are much greater than r^*, the coordinate time of (12) approaches the proper time of (8.4) as it should. In order to study the limit of $r \to r^*$, we note that the above logarithmic term can be written as

$$\ln\left|\frac{\sqrt{r}+\sqrt{r^*}}{\sqrt{r}-\sqrt{r^*}}\cdot\frac{\sqrt{r_0}-\sqrt{r^*}}{\sqrt{r_0}+\sqrt{r^*}}\right| = \ln\left|\frac{\left(\sqrt{r}+\sqrt{r^*}\right)^2}{r-r^*}\cdot\frac{r_0-r^*}{\left(\sqrt{r_0}+\sqrt{r^*}\right)^2}\right|.$$

When r is near r^*, we can drop all non-singular terms in (12) to recover the result shown in (8.4).

8.3 **Null 3-surface** We are discussing a $dr = 0$ surface. The light-like condition in the EF coordinates (*see* Table 8.1) is $ds^2 = (1 - r^*/r)dp^2 = 0$. The tangent discussed in the text holds for $d\theta = d\phi = 0$. Thus we can certainly pick the other two tangents one with $d\theta \neq 0$ and the other with $d\phi \neq 0$. With $t^\mu \equiv (t^p, t^r, t^\theta, t^\phi)$, the infinitesimal tangents $t_1^\mu = (dp, 0, 0, 0) = n^\mu$, $t_2^\mu = (dp, 0, d\theta, 0)$, $t_1^\mu = (dp, 0, 0, d\phi)$ are mutually orthogonal, with $\mathbf{t}_1 = \mathbf{n}$ being the normal to the surface as well: $g_{\mu\nu}t_1^\mu t_1^\nu = g_{\mu\nu}t_1^\mu t_2^\nu = g_{\mu\nu}t_1^\mu t_3^\nu = g_{\mu\nu}t_2^\mu t_3^\nu = 0$ because $g_{pp} = -(1 - r^*/r)$.

8.4 **Kruskal coordinates** Start from the definition $V = (p' + q')/2$, which in turn can be expressed in terms of $(\bar{t}, r)$ and $(\tilde{t}, r)$ through (8.25)

$$\begin{aligned} V &= \frac{1}{2}\left[\exp\left(p/2r^*\right) - \exp\left(-q/2r^*\right)\right] \\ &= \frac{1}{2}\left[\exp\left(\frac{\bar{t}+r}{2r^*}\right) - \exp\left(-\frac{\tilde{t}-r}{2r^*}\right)\right]. \end{aligned} \tag{13}$$

From Table 18.1 we have

$$\frac{\bar{t}+r}{2r^*} = \frac{1}{2r^*}\left[ct + r^* \ln\left|\frac{r-r^*}{r^*}\right| + r\right],$$
$$-\frac{\tilde{t}-r}{2r^*} = \frac{1}{2r^*}\left[-ct + r^* \ln\left|\frac{r-r^*}{r^*}\right| + r\right].$$

Since

$$\exp\left(\frac{1}{2}\ln\left|\frac{r-r^*}{r^*}\right|\right) = \left(\frac{r}{r^*}-1\right)^{1/2} \quad \text{for} \quad r > r^* \tag{14}$$

it then follows from (13) that

$$V = \left(\frac{r}{r^*}-1\right)^{1/2} e^{r/2r^*}\frac{e^{ct/2r^*}-e^{-ct/2r^*}}{2}$$
$$= \left(\frac{r}{r^*}-1\right)^{1/2} e^{r/2r^*}\sinh\left(\frac{ct}{2r^*}\right)$$

which is the result quoted in Eq. (8.30). The expression for $U(t,r)$ can be similarly derived. As for the regime of $r < r^*$, it just changes the sign on the RHS of (14).

8.5 **Circular orbits** For a circular orbit, the radial distance and the orbital angular momentum must satisfy a definite relation so that the effective potential (8.35)

$$\Phi_{\text{eff}} = -\frac{c^2 r^*}{2r} + \frac{l^2}{2m^2 r^2} - \frac{l^2 r^*}{2m^2 r^3}$$

is minimized (at this radial distance) $\partial\Phi_{\text{eff}}/\partial r = 0$, which fixes the angular momentum to be

$$l^2 = G_{\text{N}} M m^2 r\left(1 - \frac{3}{2}\frac{r^*}{r}\right)^{-1}. \tag{15}$$

Furthermore, $\dot{r} = 0$ for circular orbit, the total energy must equal the potential energy:

$$\mathcal{E} = m\Phi_{\text{eff}}$$

or, using the suggested form for Φ_{eff} and the relation $\mathcal{E}/m = (\kappa^2 - c^2)/2$ the total energy may be written as

$$\frac{\kappa^2 - c^2}{2} = \frac{c^2}{2}\left[\left(1 - \frac{r^*}{r}\right)\left(1 + \frac{l^2}{m^2 r^2 c^2}\right) - 1\right].$$

After plugging the result in (15), one finds

$$\kappa^2 = c^2\left(1 - \frac{r^*}{r}\right)^2\left(1 - \frac{3}{2}\frac{r^*}{r}\right)^{-1}. \tag{16}$$

From this we immediately see that at $r_0 = 3r^*$

$$[E(\infty)]_0 = mc\kappa_0 = \sqrt{\frac{8}{9}}\, mc^2.$$

8.6 **No stable circular orbit for light** Equation (7.77) can be written as $\kappa^2/2 = \dot{r}^2/2 + \Phi_{\text{eff}}$ with $\Phi_{\text{eff}}(r) = (\lambda^2/2r^2)(1-(r^*/r))$. From this one can easily show an extremum is at $r_0 = 3r^*/2$ which is unstable as $\partial^2\Phi_{\text{eff}}/\partial r^2{}_{r_0} = -2r_0^4 < 0$.

8.7 **No counter-rotating light is possible in ergosphere** The Kerr metric for an extreme spinning black hole for $dr = 0$ at $\theta = \pi/2$ is

$$ds^2 = -\left(1 - \frac{r^*}{r}\right)c^2dt^2 - \frac{r^{*2}}{r}cdtd\phi + \bar{r}^2 d\phi^2$$

where

$$\bar{r}^2 = r^2 + \frac{r^{*2}}{4} + \frac{r^{*3}}{4r}.$$

Thus the null interval $ds^2 = 0$ for a light ray obey is a quadratic equation for the angular velocity

$$\frac{\bar{r}^2}{c^2}\left(\frac{d\phi}{dt}\right)^2 - \frac{r^{*2}}{cr}\left(\frac{d\phi}{dt}\right) - \left(1 - \frac{r^*}{r}\right) = 0$$

with the solutions

$$\frac{d\phi}{dt} = \frac{cr^{*2}}{2r\bar{r}^2}\left[1 \pm \sqrt{1 + \left(\frac{2r\bar{r}^2}{cr^{*2}}\right)^2 \frac{c^2}{\bar{r}^2}\left(1 - \frac{r^*}{r}\right)}\right].$$

At the stationary limit surface (i.e. the outer boundary of the ergosphere) $r = r_S = r^*$ thus $\bar{r}^2 = (3/2)\, r^{*2}$,

$$\left[\frac{d\phi}{dt}\right]_S = \begin{cases} \frac{2c}{3r^*} & \text{co-rotating light} \\ 0 & \text{counter-rotating light.} \end{cases}$$

That is, because of frame dragging the angular velocity of counter-rotating light vanishes. At the horizon surface (i.e., the inner boundary of the ergosphere) $r = r_h = r^*/2$ thus $\bar{r} = r^*$,

$$\left[\frac{d\phi}{dt}\right]_h = \frac{c}{r^*}$$

for both co-rotating and counter-rotating light rays. Namely, inside the ergosphere, because of frame dragging, light can only rotate in the same direction as the source.

8.8 **Circulating light at the horizon** On the equatorial plane $\theta = \pi/2$ (hence $\rho = r$), the event horizon $r = r_+$ of a Kerr black hole corresponds to the $\triangle = r_+^2 - r_+ r^* + a^2 = 0$, or

$$r_+^2\left(1 - \frac{r^*}{r_+}\right) = -a^2.$$

Consequently, metric elements at r_+ have values of

$$g_{tt} = -\left(1 - \frac{r^*}{r_+}\right)c^2, \quad g_{\phi\phi} = r_+^2 + a^2\left(1 + \frac{r^*}{r_+}\right), \quad g_{t\phi} = -\frac{r^*}{r_+}ac.$$

Thus,

$$\begin{aligned} g_{tt}g_{\phi\phi} &= -r_+^2\left(1 - \frac{r^*}{r_+}\right) - \left(1 - \frac{r^{*2}}{r_+^2}\right)a^2c^2 \\ &= a^2c^2 - (1 - \frac{r^{*2}}{r_+^2})a^2c^2 = g_{t\phi}^2. \end{aligned}$$

With $g_{t\phi}^2 = g_{tt}g_{\phi\phi}$, it is clear that Eq. (8.45) has only one solution $d\phi/dt = \omega$ of (8.43) for both co-rotating and counter-rotating lights. For the extreme spinning case $r_+ = a = r^*/2$

$$\omega = -\frac{g_{t\phi}}{g_{\phi\phi}} = -\frac{-r^*c}{r^{*2}} = \frac{c}{r^*},$$

which agrees with the result shown in Problem 8.7.

8.9 **Binding energy of a particle in ISCO of a rotating black hole** Given the effective potential $\left[\Phi_{\text{eff}}^{(\text{K})}\right]$ as in (8.71) the radius can be found by the condition of $\partial\left[\Phi_{\text{eff}}^{(\text{K})}\right]/\partial r = 0$ to yield $r^2 - 2Br + C = 0$ with

$$B = \frac{l^2 - a^2m^2(\kappa^2 - c^2)}{c^2m^2r^*}, \quad C = \frac{3\left[l - am\kappa\right]^2}{c^2}.$$

The quadratic equation has the solutions $r = B \pm \sqrt{B^2 - C}$. The inner most stable orbit corresponds the case of vanishing square root, $B^2 = C$:

$$\frac{l_0^2 - a^2m^2(\kappa_0^2 - c^2)}{c^2m^2r^*} = \frac{\sqrt{3}}{c}\left[l_0 - am\kappa_0\right] \tag{17}$$

and an orbit radius of

$$r_0 = \frac{\sqrt{3}}{mc}\left[l_0 - am\kappa_0\right]. \tag{18}$$

For such orbit, the energy balance equation (as $\dot{r} = 0$) becomes

$$\frac{\kappa_0^2 - c^2}{2} = \left[\Phi_{\text{eff}}^{(\text{K})}\right]_0 = -\frac{c^2r^*}{2r_0} + \frac{l_0^2 - a^2m^2\left(\kappa_0^2 - c^2\right)}{2m^2r_0^2} - \frac{r^*\left(l_0 - am\kappa_0\right)^2}{2m^2r_0^3}. \tag{19}$$

978–0–19–957364–6 17-Cheng-AppB Cheng (Typeset by SPi, Chennai) 391 of 414 September 24, 2009 17:51

We thus have three equations (17), (18), and (19) for the three unknowns of (r_0, l_0, κ_0). It is straightforward exercise to check that $r_0 = r^*/2,\ l_0 = mcr^*/\sqrt{3},\ \kappa_0 = c/\sqrt{3}$ satisfy these equations for an extreme spinning black hole $(a = r^*/2)$. This is the result of an extraordinary binding energy of $0.42\, mc^2$.

9.2 **Luminosity distance to the nearest star** The observed flux being $f = \mathfrak{L}/4\pi d^2$, we have

$$d_* = \left(\frac{f_\odot}{f_*}\right)^{1/2} \times \mathrm{AU} = 3 \times 10^5 \mathrm{AU} = 1.5\ \mathrm{pc}.$$

9.3 **Gravitational frequency shift contribution to the Hubble redshift** The gravitational redshift being given by (4.26), we can estimate to be

$$z_\mathrm{G} = \frac{M_\mathrm{G}}{M_\odot}\frac{R_\odot}{R_\mathrm{G}} z_\odot = O(10^{-7}),$$

Thus, the shift due to gravity is quite negligible.

9.4 **Energy content due to star light** Let us denote the average stellar luminosity by $\mathfrak{L}_*$ and star number density by n. Their product is then the luminosity density as given by (9.19),

$$n\mathfrak{L}_* = 2 \times 10^8 \frac{\mathfrak{L}_\odot}{(\mathrm{Mpc})^3} = 2.6 \times 10^{-33}\ \mathrm{W\ m^{-3}},$$

which is the energy emitted per unit volume per unit time. Stars have been assumed to be emitting light at this luminosity during the entire $t_0 \simeq t_\mathrm{H} \simeq 13.6$ Gyr $= 4.3 \times 10^{17}$ s, leading to an energy density contribution at present of $\rho_* c^2 = n\mathfrak{L}_* t_\mathrm{H} \simeq 10^{-15}\ \mathrm{J\,m^{-3}}$ or, using (9.17), a density ratio $\Omega_* = (\rho_*/\rho_\mathrm{c}) \simeq 10^{-5}$.

9.5 **Night sky as bright as day** Flux being in watts per unit area, the total flux due to all the starlights is, according to (9.2) and Problem 9.4,

$$f_* = (n\mathfrak{L}_*)\, ct_H \simeq \left(2 \times 10^8 \frac{\mathfrak{L}_\odot}{\mathrm{Mpc}^3}\right) ct_H$$
$$= 0.8 \times 10^{12} \frac{\mathfrak{L}_\odot}{\mathrm{Mpc}^2} = 2.5 \times 10^{-10} \frac{\mathfrak{L}_\odot}{4\pi(\mathrm{AU})^2}.$$

Thus, we need to lengthen the age by a factor of 4 billion before we can get a night sky as bright as day!

9.6 **The Virial theorem** Time derivative of the virial yields (for notational simplicity we drop subscript n and the summation sign)

$$\frac{dG}{dt} \equiv \mathbf{p}\cdot\dot{\mathbf{r}} + \dot{\mathbf{p}}\cdot\mathbf{r} = mv^2 + \mathbf{F}\cdot\mathbf{r} = 2T - \frac{\partial V}{\partial r} r = 2T + V$$

where, to reach the last expression, we have used the r dependence of the gravitational potential $V = ar^{-1}$. We now time average $\langle..\rangle = \tau^{-1}\int dt$ this equation; the LHS becomes

$$\left\langle\frac{dG}{dt}\right\rangle = \frac{1}{\tau}\int_0^\tau dG = \frac{1}{\tau}\left[G(\tau) - G(0)\right]$$

which vanishes for a periodic system. In this way we obtain the virial theorem of $2\langle T\rangle + \langle V\rangle = 0$.

9.8 **Wavelength in an expanding universe** A radial light signal follows the null worldline in the RW geometry and its proper distance is given by (9.45). Consider two successive wavecrests with wavelength λ; the second one is emitted (and recieved) later by a time interval $\delta t = \lambda/c$. Both wavecrests travel the same distance $d_{\rm p}(\xi, t_0)$:

$$(d_{\rm p} =)\quad \int_{t_{\rm em}}^{t_0} \frac{cdt}{a(t)} = \int_{t_{\rm em}+\lambda_{\rm em}/c}^{t_0+\lambda_0/c} \frac{cdt}{a(t)}.$$

After cancelling out the common interval from $t_{\rm em} + \lambda_{\rm em}/c$ to t_0 from both sides of the integral equality, we have

$$\int_{t_{\rm em}}^{t_{\rm em}+\lambda_{\rm em}/c} \frac{cdt}{a(t)} = \int_{t_0}^{t_0+\lambda_0/c} \frac{cdt}{a(t)}.$$

Since the scale factor would not have changed much during the small time interval between these two crests

$$\frac{1}{a(t_{\rm em})}\int_{t_{\rm em}}^{t_{\rm em}+\lambda_{\rm em}/c} dt = \frac{1}{a(t_0)}\int_{t_0}^{t_0+\lambda_0/c} dt$$

which immediately leads to the expected result of $(\lambda_0/\lambda_{\rm em}) = a(t_0)/a(t_{\rm em})$.

9.9 **The deceleration parameter and Taylor expansion of the scale factor**

$$\begin{aligned} a(t) &\simeq a(t_0) + (t-t_0)\dot a(t_0) + \frac{1}{2}(t-t_0)^2\ddot a(t_0) \\ &= 1 + (t-t_0)H_0 - \frac{1}{2}(t-t_0)^2 q_0 H_0^2 \end{aligned} \tag{20}$$

and

$$\frac{1}{a(t)} \simeq 1 - (t-t_0)H_0 + (t-t_0)^2\left(1+\frac{q_0}{2}\right)H_0^2. \tag{21}$$

9.10 **The steady-state universe**

(a) "Perfect CP" means that the universe is not only homogeneous in space but also in time.

(b) From (9.43) we have $da/dt = H_0 a$, which has the solution $a(t) = \exp H_0(t-t_0)$. Thus $\dot a = H_0 a$ and $\ddot a = H_0^2 a$ so that $q_0 = -1$.

(c) According to (5.46), the curvature for the 3D space in the Steady-State Universe (SSU) is $K = kR^{-2}(t)$. Since the scale factor does depend on t, an unchanging K can come about only for the curvature signature $k = 0$. Namely, an SSU requires a 3D space with a flat geometry.

(d) For a constant density, the rate of mass increase must be proportional to that of volume increase

$$\frac{dM}{dt} = \frac{dM}{dV}\frac{dV}{dt} = \rho_{\rm M}\frac{dV}{dt},$$

while the normalized volume increase rate can be directly related to the Hubble constant

$$\frac{\dot{V}}{V} = \frac{3\dot{a}}{a} = 3H.$$

The mass creation rate per unit volume can then be calculated

$$\frac{\dot{M}}{V} = \rho_{\rm M}\frac{\dot{V}}{V} = 3H_0\rho_{\rm M} \simeq 0.7 \times 10^{-24}\ \text{g/year/km}^3.$$

Given that $m_{\rm p} = 1.7 \times 10^{-24}$ g, this means the creation of one hydrogen atom, in a cubic kilometer volume, every 2–3 years.

9.11 **z^2 correction to the Hubble relation**

(a) From (9.45)

$$d_{\rm p}(t_0) = a(t_0)\int_{t_{\rm em}}^{t_0}\frac{cdt}{a(t)}$$

and the first two terms of the Taylor series (21) we have

$$d_{\rm p}(t_0) = c(t_0 - t_{\rm em}) + \frac{c}{2}H_0(t_0 - t_{\rm em})^2. \tag{22}$$

The first term on the RHS is just the distance traversed by a light signal in a static environment; the second term represents the correction due to the expansion of the universe.

(b) $(t_0 - t_{\rm em})$ can be related to the redshift z through (9.50) and (21):

$$z = -1 + \frac{1}{a(t_{\rm em})} = (t_0 - t_{\rm em})H_0\left[1 + (t_0 - t_{\rm em})H_0\left(1 + \frac{q_0}{2}\right)\right]. \tag{23}$$

(c) Equation (23) can be inverted to yield

$$\begin{aligned} t_0 - t_{\rm em} &\simeq \frac{z}{H_0}\left[1 - (t_0 - t_{\rm em})H_0\left(1 + \frac{q_0}{2}\right)\right] \\ &\simeq \frac{z}{H_0}\left[1 - z\left(1 + \frac{q_0}{2}\right)\right]. \end{aligned} \tag{24}$$

Plug this expression for the look-back time into (22), we have

$$\begin{aligned} D_{\rm p}(t_0) &\simeq \left[\frac{cz}{H_0} - \frac{cz^2}{H_0}\left(1 + \frac{q_0}{2}\right)\right] + \frac{cz^2}{2H_0} \\ &= \frac{cz}{H_0}\left(1 - \frac{1+q_0}{2}z\right). \end{aligned}$$

10.2 **Newtonian interpretation of second Friedmann equation** For the pressureless matter used for our Newtonian system, cf. Fig. 10.1, the gravitational attraction by the whole sphere being $-G_{\rm N}M/r^2 = \ddot{r}$, or $-(4\pi/3)G_{\rm N}a\rho = \ddot{a}$, which is just Eq. (10.2) without the pressure term.

10.4 **The empty universe** The nontrivial solution to $\dot{a}^2 = -kc^2R_0^{-2}$ is a negatively curved open universe $k = -1$ and $a = t/t_0$, with $t_0 = c^{-1}R_0$, which is just the straight-line $a(t)$ in Fig. 10.2. From (9.45) we can obtain the proper distance in terms of z.

$$d_{\rm p}(t_0) = \int_{t_{\rm em}}^{t_0} \frac{c\,dt}{a(t)} = ct_0 \int_{t_{\rm em}}^{t_0} t^{-1} dt = ct_0 \ln\left(\frac{t_0}{t_{\rm em}}\right) = ct_0 \ln(1+z),$$

where we have used $t_0/t_{\rm em} = (a(t_{\rm em}))^{-1} = (1+z)$. It is clear that for small redshift this equation reduces to the Hubble relation (9.5) with $H_0 = t_0^{-1}$. Namely, in an empty universe the age is given by the Hubble time $t_0 = t_{\rm H}$, and the "radius" by the Hubble length $R_0 = l_{\rm H} = ct_{\rm H}$.

10.5 **Hubble plot in matter-dominated flat universe** Since distance modulus is a simple logarithmic expression (9.62) of luminosity distance $d_{\rm L}$, which in turn is related to our proper distance $d_{\rm p}$ by $d_{\rm L} = (1+z)d_{\rm p}$ of (9.57), all we need is to calculate the proper distance according to (9.47). This integration can be performed for this matter dominated flat universe, which has $a(t) = (t/t_0)^{2/3}$ as given in (10.30):

$$\begin{aligned} d_{\rm p}(t_0) &= \int_{t_{\rm em}}^{t_0} \frac{c\,dt}{a(t)} = ct_0^{2/3} \int_{t_{\rm em}}^{t_0} t^{-2/3} dt = 3ct_0 \left[1 - \left(\frac{t_{\rm em}}{t_0}\right)^{1/3}\right] \\ &= 3ct_0 \left(1 - [a\,(t_{\rm em})]^{1/2}\right) = \frac{2c}{H_0}(1 - 1 + z^{-1/2}), \end{aligned}$$

where we have used $a(t_{\rm em}) = (t_{\rm em}/t_0)^{2/3}$ one more time as well as the basic redshift relation of (9.50) and the age of a flat MDU $t_0 = \frac{2}{3}H_0^{-1}$ of (10.30). This way one finds the distance modulus to be

$$m - M = 5\log_{10} \frac{2cH_0^{-1}(1 + z - 1 + z^{1/2})}{10\ \text{pc}}.$$

10.7 **Distance to a light emitter at redshift z** Plug (10.26), $a(t) = (t/t_0)^x$ into Eq. (9.47), we have

$$d_{\rm p}(t_0) = \int_{t_{\rm em}}^{t_0} \frac{c\,dt'}{a(t')} = \frac{ct_0}{1-x}\left[1 - \left(\frac{t_{\rm em}}{t_0}\right)^{1-x}\right]. \tag{25}$$

On the other hand, the time of light emission from a receding galaxy with redshift z can be obtained through (9.50) and (10.26)

$$1 + z = \frac{a(t_0)}{a(t_{\rm em})} = \left(\frac{t_0}{t_{\rm em}}\right)^x$$

and thus $t_{\rm em} = t_0(1+z)^{-1/x}$. Plugging into (25), we have

$$d_{\rm p}(t_0) = \frac{ct_0}{1-x}\left[1 - \frac{1}{(1+z)^{(1-x)/x}}\right].$$

We note in particular for a matter-dominated flat universe $x = \frac{2}{3}$ we have

$$d_{\rm p}(t_0) = 3ct_0\left[1 - \frac{1}{(1+z)^{1/2}}\right], \tag{26}$$

which for $t_0 = 2/(3H_0)$ agrees with the result obtained in Problem 10.5. For a radiation-dominated flat universe $x = \frac{1}{2}$, we have $d_p(t_0) = 2ct_0 1 - (1+z)^{-1}$. NB: These simple relations between redshift and time hold only for a universe with a single-component on energy content; moreover, it does not apply to the situation when the equation-of-state parameter is negative ($w = -1$), even though the energy content is a single-component case.

10.8 **Scaling behavior of number density and Hubble's constant**

(a) For material particles the number density scales as the inverse volume factor, $n(t)/n_0 = a(t)^{-3}$. The basic relation (9.50) between scale factor and redshift leads to $n(t)/n_0 = (1+z)^3$. This scaling property also holds for radiation because $n \sim T^3 \sim a^{-3}$ as given in (10.35).

(b) We can obtain the scaling behavior of the Hubble parameter from Friedmann equation (10.1) for a flat universe: $\dot{a}^2/a^2 = 8\pi G_N \rho/3$, which can be written as $H^2/H_0^2 = \rho/\rho_{c,0}$. For an epoch when the density is dominated by radiation $\rho \simeq \rho_R = \rho_{R,0} a^{-4}$, the above expression for H becomes $H^2/H_0^2 = \Omega_{R,0}(1+z)^4$. Similarly, in a matter dominated epoch obeys $H^2/H_0^2 = \Omega_{M,0}(1+z)^3$.

10.9 **Radiation and matter equality time** Since the universe from t_{RM} to t_γ is matter-dominated, we have from (10.30), $a(t_{RM})/a(t_\gamma) = (t_{RM}/t_\gamma)^{2/3}$, or

$$t_{RM} = \left[\frac{a(t_{RM})}{a(t_\gamma)}\right]^{3/2} t_\gamma = \left[\frac{1+z_{RM}}{1+z_\gamma}\right]^{-3/2} t_\gamma \simeq \frac{t_\gamma}{8^{3/2}} \simeq 16\,000 \text{ years.}$$

10.10 **Density and deceleration parameter** Use the definition of w in (10.4), the second Friedmann Eq. (10.2) becomes

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G_N}{3} \sum_i \rho_i (1 + 3w_i).$$

In terms of the deceleration parameter (9.63) $q_0 \equiv -\ddot{a}(t_0)/a(t_0)H_0^2$ and the critical density (10.6) the second derivative equation leads to the claimed result

$$q_0 = \frac{1}{2}\sum_i \Omega_{i,0}(1+3w_i) = \Omega_{R,0} + \frac{1}{2}\Omega_{M,0} + \cdots$$

10.13 **Cosmological limit of neutrino mass** Even if we assume that all the non-baryonic dark matter is made of three species (flavors) of neutrinos $\rho_{DM} = \Sigma_{i=1}^3 \rho(\nu_i) = 3n_\nu \bar{m}$, where n_ν is the neutrino number density and $\bar{m}$ is the average neutrino mass. From the neutrino and photon temperature of (10.75) and density being the cubic power of temperature (10.35),

$$n_\nu = \left(\frac{T_\nu}{T_\gamma}\right)^3 n_\gamma \simeq (1.7)^{-3} \times 400 \simeq 150 \text{ cm}^{-3}.$$

The energy density ratio becomes $\Omega_{DM} = (3n_\nu \bar{m}c^2)/(\rho_c c^2) \simeq 0.21$. Using the critical energy density value of (9.17), we have the upper limit for the averaged neutrino mass of $\bar{m}c^2 \simeq (0.21 \times 5500)/(3 \times 150) \simeq 3$ eV.

10.14 **Temperature dipole anisotropy as Doppler effect** Recall that temperature scales as a^{-1}, that is, as inverse wavelength, or as frequency: $\delta T/T = \delta\omega/\omega$.

But the nonrelativistic Doppler effect (the small β limit of (3.46)) reads $\omega' = (1 - (v/c)\cos\theta)\omega$ or $(\delta\omega/\omega) = (v/c)\cos\theta$.

11.1 **Another form of the expansion equation** Consider the energy balance equation (10.11), $\dot{r}^2/2 - G_N M/r = \text{const.}$ leading to $\dot{a}^2 - (8\pi/3)\ G_N \rho a^2 = \text{const}'$. which can also be obtained easily from (10.1). Dividing through by the second term and using the definition of critical density we have $\Omega^{-1} - 1 = \text{const.}/(\rho a^2)$.

11.2 **The epoch-dependent Hubble constant and $a(t)$** Using (10.7) to replace the curvature parameter k in the Friedmann equation (10.1), we have

$$\frac{\dot{a}^2(t)}{a^2(t)} = \frac{8\pi G_N}{3}\rho + \frac{\dot{a}^2(t_0)}{a^2(t)}(1-\Omega_0) = H_0^2\left(\frac{\rho}{\rho_{c,0}} + \frac{1-\Omega_0}{a^2(t)}\right). \quad (27)$$

Putting the time-dependence of the densities

$$\frac{\rho}{\rho_{c,0}} = \Omega(t) = \frac{\Omega_{R,0}}{a^4} + \frac{\Omega_{M,0}}{a^3} + \Omega_{\Lambda,0},$$

Eq. (27) becomes

$$\frac{H^2(t)}{H_0^2} = \frac{\Omega_{R,0}}{a^4} + \frac{\Omega_{M,0}}{a^3} + \Omega_{\Lambda,0} + \frac{1-\Omega_0}{a^2}.$$

11.4 **Negative Λ and the "big crunch"** For the $\Omega_0 = 1$ flat universe with matter and dark energy, we have the Friedmann equation (11.38)

$$H(a) = H_0 \Omega_{M,0} a^{-3} + \Omega_\Lambda{}^{1/2}.$$

At $a = a_{\max}$ the universe stops expanding and $H(a_{\max}) = 0$, thus $a_{\max} = (-\Omega_{M,0}/\Omega_\Lambda)^{1/3}$. The cosmic time for the big crunch being twice the time for the universe to go from $a_{\max}$ to $a = 0$, we calculate in a way similar to that shown in sidenote 28,

$$2t_H \int_0^{a_{\max}} \frac{da}{\Omega_{M,0}a^{-1} + \Omega_\Lambda a^2{}^{1/2}} = \frac{4t_H}{3\sqrt{-\Omega_\Lambda}} \int_0^{a_{\max}^{3/2}} \frac{dx}{a_{\max}^3 - x^2{}^{1/2}}$$

$$= \frac{4t_H}{3\sqrt{-\Omega_\Lambda}}\left[\sin^{-1}\left(\frac{x}{a_{\max}^{3/2}}\right)\right]_0^{a_{\max}^{3/2}} = \frac{2\pi}{3\sqrt{-\Omega_\Lambda}} t_H = t_*.$$

11.5 **Estimate of matter and dark energy equality time** We define the matter and dark energy equality time $t_{M\Lambda}$ as $\rho_M(t_{M\Lambda}) = \rho_\Lambda(t_{M\Lambda})$. Using the scaling properties of these densities we have $\rho_{M,0}/a_{M\Lambda}^3 = \rho_{\Lambda,0}$ or $1 + z_{M\Lambda} = (a_{M\Lambda})^{-1} = (\Omega_\Lambda/\Omega_{M,0})^{1/3}$, which differs from the result in (11.48) by a factor of $2^{1/3} \approx 1.25$ to yield $a_{M\Lambda} = 0,7$ (and a redshift of $z_{M\Lambda} = 0.42$). Using the formula given in sidenote 25, we obtain the corresponding cosmic age $t_{M\Lambda} = t(a_{M\Lambda}) = 9.5$ Gyr.

12.1 **Basis and inverse basis vectors: a simple exercise**

(a) Given the basis vectors, the inverse basis vectors can be worked out: $\mathbf{e}^1 = \frac{1}{a}(1, -\cot\theta)$ and $\mathbf{e}^2 = \frac{1}{b}(0, \csc\theta)$. The condition $\mathbf{e}_1 \cdot \mathbf{e}^1 = \mathbf{e}_2 \cdot \mathbf{e}^2 = 1$ and $\mathbf{e}_1 \cdot \mathbf{e}^2 = \mathbf{e}_2 \cdot \mathbf{e}^1 = 0$ can be easily checked by explicit vector multiplication. For example, $\mathbf{e}_2 \cdot \mathbf{e}^1 = \frac{b}{a}(\cos\theta - \cos\theta) = 0$.

(b) Similarly by explicit vector multiplications, we have

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j = \begin{pmatrix} a^2 & ab\cos\theta \\ ab\cos\theta & b^2 \end{pmatrix}$$

$$g^{ij} = \mathbf{e}^i \cdot \mathbf{e}^j = \begin{pmatrix} \frac{1}{a^2\sin^2\theta} & -\frac{\cos\theta}{ab\sin^2\theta} \\ -\frac{\cos\theta}{ab\sin^2\theta} & \frac{1}{b^2\sin^2\theta} \end{pmatrix}$$

so that $g_{ij}g^{jk} = \delta_{ik}$ can be checked by matrix multiplication.

(c) We can verify the completeness condition by calculating the direct-products of basis vectors,

$$\sum_i \mathbf{e}_i \otimes \mathbf{e}^i = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & -\cot\theta \end{pmatrix} + \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \begin{pmatrix} 0 & \csc\theta \end{pmatrix}$$
$$= \begin{pmatrix} 1 & -\cot\theta \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \cot\theta \\ 0 & 1 \end{pmatrix} = \mathbf{1}.$$

12.4 **Transformation: coordinates vs. basis vectors** A^μ transform "oppositely" from the bases vectors $\mathbf{e}_\mu$

$$\mathbf{e}_\mu \longrightarrow \mathbf{e}'_\mu = \left[\mathbf{L}^{-1}\right]_\mu^{\ \nu} \mathbf{e}_\nu \tag{28}$$

because the vector itself $\mathbf{A} = A^\mu \mathbf{e}_\mu$ does not change under the coordinate transformations.

12.5 **$g_{\mu\nu}$ is a tensor**

(a) Plugging in the transformations of the basis vectors (28) in the metric definition $g'_{\mu\nu} = \mathbf{e}'_\mu \cdot \mathbf{e}'_\nu$ we immediately obtain that for the metric, (12.17).

(b) The invariance of the scalar product $\mathbf{A} \cdot \mathbf{B}$ can also expressed as

$$A_\mu B_\nu g^{\mu\nu} = A'_\lambda B'_\rho g'^{\lambda\rho} = A_\mu B_\nu \left[\mathbf{L}^{-1}\right]^\mu_{\ \lambda} \left[\mathbf{L}^{-1}\right]^\nu_{\ \rho} g'^{\lambda\rho},$$

or

$$g^{\mu\nu} = \left[\mathbf{L}^{-1}\right]^\mu_{\ \lambda} \left[\mathbf{L}^{-1}\right]^\nu_{\ \rho} g'^{\lambda\rho}.$$

We can invert this equation by multiplying two [**L**] factors on both sides to obtain

$$g'^{\mu\nu} = [\mathbf{L}]^\mu_{\ \lambda} [\mathbf{L}]^\nu_{\ \rho} g^{\lambda\rho}.$$

This shows, cf. Eq. (12.16), that the (inverse) metric is indeed a *bona fide* contravariant tensor.

12.6 **The quotient theorem** Given that the product $A^\mu B^\nu g_{\mu\nu}$ is a scalar, and vectors A^μ and B^ν are known to be tensors, their quotient $g_{\mu\nu}$ must also be a tensor.

12.7 **Transformation of the metric** From (5.34) we can immediately deduce the coordinate transformation

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{pmatrix} \begin{pmatrix} dr \\ d\phi \end{pmatrix},$$

hence the transformation matrix

$$[\mathbf{L}] = \begin{pmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{pmatrix} \text{ and } \left[\mathbf{L}^{-1}\right] = \begin{pmatrix} \cos\phi & \sin\phi \\ -r^{-1}\sin\phi & r^{-1}\cos\phi \end{pmatrix}.$$

Clearly the metric transformation checks out

$$\left[\mathbf{L}^{-1}\right]^{\mathsf{T}} \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \left[\mathbf{L}^{-1}\right] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

12.8 **Generalized orthogonality condition and the boost transformation** We can work this out in a way that's entirely similar to Problem 3.4. Writing out the condition $\mathbf{L}\eta\mathbf{L}^{\mathsf{T}} = \eta$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -1 & 0 \\ & 0\ 1 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ & 0\ 1 \end{pmatrix}, \tag{29}$$

we have the conditions of $a^2 - b^2 = -c^2 + d^2 = 1$, which can be solved by the parametrization of $a = \cosh\psi,\ b = \sinh\psi$ and $c = \sinh\psi',\ d = \cosh\psi'$, while the off-diagonal condition of $-ac + bd = -\cosh\psi\sinh\psi' + \sinh\psi\cosh\psi' = \sinh(\psi - \psi') = 0$ yields $\psi = \psi'$. The identification of $\tanh\psi = v/c$ was worked out in Box 3.2.

12.9 **Covariant Lorentz force law**

(a) This identification is justified because our relativistic force $\vec{F} = \gamma m\vec{a}$ becomes the usual $\vec{F} = m\vec{a}$ in the non-relativistic situation when $\gamma = 1$.

(b) For $\mu = i$,

$$K^i = \frac{q}{c}F^{i\nu}U_\nu = \frac{q}{c}\left(F^{i0}U_0 + F^{ij}U_j\right),$$
$$\gamma F_i = \frac{q}{c}\left[-E_i\,(-\gamma c) + \epsilon_{ijk}B_k\left(\gamma v_j\right)\right],$$

which, after the cancellation of the γ factor from both sides, is just the familiar Lorentz force law written in its components.

(c) For $\mu = 0$,

$$K^0 = \frac{q}{c}F^{0i}U_i = \gamma\frac{q}{c}\vec{E}\cdot\vec{v} \tag{30}$$

is indeed $\gamma\vec{F}\cdot\vec{v}/c$ because the dot product with the magnetic field term in Lorentz force vanishes.

12.11 **Homogeneous Maxwell's equations** To show that $\partial_\mu F_{\nu\lambda} + \partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} = 0$ follows from $\partial_\mu\tilde{F}^{\mu\nu} = 0$: From the definition of dual field tensor, we have $\partial_\mu F_{\lambda\rho}\epsilon^{\mu\nu\lambda\rho} = 0$, which is a trivial relation $(0 = 0)$ if any pair of indices in (μ, λ, ρ) are equal. Thus, only when the indices are unequal do we get non-trivial relation: take the example of equation of $\partial_\mu \tilde{F}^{\mu 0} = \partial_\mu F_{\lambda\rho}\epsilon^{\mu 0\lambda\rho} = 0$ we have

$$\partial_1 F_{23} + \partial_3 F_{12} + \partial_2 F_{31} = 0.$$

We can regard this as a relation in a particular coordinate frame with $\mu = 1$, $\nu = 2$, and $\lambda = 3$. Once written in the Lorentz covariant version, it must be valid in every frame. This is just the relation we set out to prove:

$$\partial_\mu F_{\nu\lambda} + \partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} = 0.$$

To prove the converse statement, all we need to do is to contract $\epsilon^{\mu\nu\lambda\rho}$ onto the above equation.

12.15 **$T^{\mu\nu}$ for a system of EM field and charges** We first calculate the divergence of $T^{\mu\nu}_{\text{charge}} = \rho'_{\text{mass}} U^\mu U^\nu$ to find that

$$\partial_\mu T^{\mu\nu}_{\text{charge}} = \rho'_{\text{mass}} \left(U^\mu \partial_\mu\right) U^\nu$$

where we have also used the mass conservation law of $\partial_\mu(\rho'_{\text{mass}} U^\mu) = 0$. The Lorentz invariant product $U^\mu \partial_\mu$ can be evaluated in any convenient reference frame; we choose the comoving frame $U^\mu = \gamma(c, \vec{0})$ to obtain $U^\mu \partial_\mu = \gamma \partial_t = \partial_\tau$, the differentiation with respect to the proper time τ. The term $\rho'_{\text{mass}} \partial_\tau U^\nu$ is the 4-force density (ie, mass replace by mass density). Use the formula (12.44) for the Lorentz force density (charge replaced by charge density), we then have

$$\partial_\mu T^{\mu\nu}_{\text{charge}} = \rho'_{\text{mass}} \partial_\tau U^\nu = \frac{\rho'_{\text{charge}}}{c} F^{\nu\lambda} U_\lambda = \frac{1}{c} F^{\nu\lambda} j_\lambda$$

where we have used the expression for the electromagnetic current for free charges $j_\lambda = \rho'_{\text{charge}} U_\lambda$.

We now calculate the divergence of $T^{\mu\nu}_{\text{field}}$ in (12.80) to find

$$\partial_\mu T^{\mu\nu}_{\text{field}} = \eta_{\alpha\beta} \left(\partial_\mu F^{\mu\alpha}\right) F^{\nu\beta}.$$

Here we have used the calculation performed in (12.82) and by noting the fact that, in the presence of charges, the inhomogeneous Maxwell's equation $\partial_\mu F^{\mu\alpha} = -\frac{1}{c} j^\alpha$ has a non-vanishing RHS

$$\partial_\mu T^{\mu\nu}_{\text{field}} = -\frac{1}{c} \eta_{\alpha\beta} j^\alpha F^{\nu\beta} = -\frac{1}{c} F^{\nu\lambda} j_\lambda.$$

This shows clearly that the sum $T^{\mu\nu} = T^{\mu\nu}_{\text{field}} + T^{\mu\nu}_{\text{charge}}$ has zero divergence.

12.16 **Radiation pressure and energy density** The system of electromagnetic field can be viewed either as a system of field with energy–momentum tensor

$$T^{\mu\nu}_{\text{field}} = \eta_{\alpha\beta} F^{\mu\alpha} F^{\nu\beta} - \frac{1}{4} \eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta},$$

or as a system of ideal fluid made up of photons with, cf. (12.72),

$$T^{\mu\nu}_{\gamma\,\text{fluid}} = \begin{pmatrix} \rho' c^2 & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix}$$

with $\rho' c^2$ and p being the radiation energy density and pressure, respectively. Since these two representations both describe the same system we should expect $T^{\mu\nu}_{\gamma\,\text{fluid}} = T^{\mu\nu}_{\text{field}}$, in particular their traces should equal: $\eta_{\mu\nu} T^{\mu\nu}_{\gamma\,\text{fluid}} = \eta_{\mu\nu} T^{\mu\nu}_{\text{field}}$. But a simple inspection shows that $\eta_{\mu\nu} T^{\mu\nu}_{\text{field}} = 0$ because $\eta_{\mu\nu}\eta^{\mu\nu} = 4$. The vanishing trace $\eta_{\mu\nu} T^{\mu\nu}_{\gamma\,\text{fluid}} = 0$ leads to the result $p = \rho' c^2/3$. (That $T^{\mu\nu}$ is traceless is related to the scale invariance of the system.)

13.1 **Covariant derivative for covariant components** Given that $A_\mu A^\mu$ is an invariant, in the notation of (13.43), we also have $\Delta(A_\mu A^\mu)_{\text{coord}} = 0$:

$$A_\mu \left[\Delta A^\mu\right]_{\text{coord}} + A^\mu \left[\Delta A_\mu\right]_{\text{coord}} = 0.$$

$\Delta A^\mu{}_{\text{coord}}$ being given by (13.45), we get

$$A^\mu \left[\Delta A_\mu\right]_{\text{coord}} = A_\mu \Gamma^\mu_{\nu\lambda} A^\nu dx^\lambda = A^\mu \left(\Gamma^\nu_{\mu\lambda} A_\nu dx^\lambda\right).$$

The last expression is reached by relabelling $\mu \leftrightarrow \nu$. The result of $\Delta A_{\mu\,\text{coord}} = +\Gamma^\nu_{\mu\lambda} A_\nu dx^\lambda$ implies that $D_\nu A_\mu = \partial_\nu A_\mu - \Gamma^\lambda_{\nu\mu} A_\lambda$.

13.2 **Christoffel symbols of polar coordinates for a flat plane**

(a) Explicitly differentiating the relation $\mathbf{r} = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j}$, we have

$$d\mathbf{r} \equiv dr\,\mathbf{e}_r + d\theta\,\mathbf{e}_\theta = dr\cos\theta\,\mathbf{i} - r\sin\theta d\theta\,\mathbf{i} + dr\sin\theta\,\mathbf{j} + r\cos\theta d\theta\,\mathbf{j}.$$

Collecting the dr and $d\theta$ terms,

$$\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}, \quad \mathbf{e}_\theta = -r\sin\theta\,\mathbf{i} + r\cos\theta\,\mathbf{j}.$$

The inverse bases can be gotten by contracting with the inverse metric $g^{\mu\nu} = \text{diag}(1,\ r^{-2})$:

$$\mathbf{e}^r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}, \quad \mathbf{e}^\theta = -r^{-1}\sin\theta + r^{-1}\cos\theta\,\mathbf{j}.$$

(b) To calculate the Christoffel symbols through their definition of $\partial_\nu \mathbf{e}^\mu = -\Gamma^\mu_{\nu\lambda}\mathbf{e}^\lambda$ we first observe:

$$\frac{\partial \mathbf{e}^r}{\partial r} = 0, \quad \frac{\partial \mathbf{e}^\theta}{\partial r} = \frac{-1}{r^2}\left(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}\right) = \frac{-1}{r}\mathbf{e}^\theta.$$

Then the definitions

$$\frac{\partial \mathbf{e}^r}{\partial r} = \Gamma^r_{rr}\mathbf{e}^r + \Gamma^r_{r\theta}\mathbf{e}^\theta, \quad \frac{\partial \mathbf{e}^\theta}{\partial r} = \Gamma^\theta_{rr}\mathbf{e}^r + \Gamma^\theta_{r\theta}\mathbf{e}^\theta$$

allow us to read off the Christoffel symbols $\Gamma^r_{rr} = \Gamma^r_{r\theta} = \Gamma^\theta_{rr} = 0$ and $\Gamma^\theta_{r\theta} = r^{-1}$. Similarly, from

$$\frac{\partial \mathbf{e}^r}{\partial \theta} = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j} = r\mathbf{e}^\theta,$$

$$\frac{\partial \mathbf{e}^\theta}{\partial \theta} = -r^{-1}\cos\theta\,\mathbf{i} - r^{-1}\sin\theta\,\mathbf{j} = -r^{-1}\mathbf{e}^r$$

we obtain $\Gamma^r_{\theta r} = \Gamma^\theta_{\theta\theta} = 0$, $\Gamma^r_{\theta\theta} = -r$ and $\Gamma^\theta_{\theta r} = r^{-1}$.

(c) Work out the components in

$$\begin{aligned} D_\mu A^\mu &= \partial_\mu A^\mu + \Gamma^\mu_{\mu\nu} A^\nu \\ &= \partial_r A^r + \partial_\theta A^\theta + \left(\Gamma^r_{rr} + \Gamma^\theta_{\theta r}\right) A^r + \left(\Gamma^r_{r\theta} + \Gamma^\theta_{\theta\theta}\right) A^\theta \\ &= \partial_r A^r + \partial_\theta A^\theta + \frac{1}{r} A^r = \frac{1}{r}\frac{\partial}{\partial r}\left(r A^r\right) + \frac{\partial}{\partial \theta} A^\theta \\ &= \begin{pmatrix} \frac{1}{r}\frac{\partial}{\partial r} r & \frac{\partial}{\partial \theta} \end{pmatrix} \begin{pmatrix} A^r \\ A^\theta \end{pmatrix}. \end{aligned}$$

(d) Because the scalar function $\Phi(x)$ is coordinate independent, $D_\mu \Phi = \partial_\mu \Phi$. To raise the index we must multiply it by the inverse metric $g^{\mu\nu}\partial_\mu \Phi$. Using the result obtained in (c) we have

$$\begin{aligned} D_\mu D^\mu \Phi(x) &= D_\mu \left(g^{\mu\nu}\partial_\mu \Phi\right) \\ &= \begin{pmatrix} \frac{1}{r}\frac{\partial}{\partial r} r & \frac{\partial}{\partial \theta} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & r^{-2} \end{pmatrix} \begin{pmatrix} \partial_r \Phi \\ \partial_\theta \Phi \end{pmatrix} \\ &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2}. \end{aligned}$$

(e) The metric in polar coordinates has only one nontrivial element $g_{\theta\theta} = r^2$. Checking the covariant differentiation with respect to the radial coordinate r, we get

$$D_r g_{\theta\theta} = \partial_r g_{\theta\theta} - 2\Gamma^\mu_{r\theta} g_{\mu\theta} = 2r - 2\frac{1}{r}r^2 = 0.$$

(f) Substituting $g_{\theta r} = 0$ and $g_{\theta\theta} = r^2$ into (13.37), we have

$$\begin{aligned} \Gamma^r_{\theta\theta} &= \frac{1}{2} g^{r\mu}\left(\partial_\theta g_{\theta\mu} + \partial_\theta g_{\theta\mu} - \partial_\mu g_{\theta\theta}\right) \\ &= \frac{1}{2} g^{rr}\left(2\partial_\theta g_{\theta r} - \partial_r g_{\theta\theta}\right) = -r \\ \Gamma^\theta_{\theta\theta} &= \frac{1}{2} g^{\theta\theta}\partial_\theta g_{\theta\theta} = 0. \end{aligned}$$

(g) We have a diagonal metric $g_{11} = g_{rr} = 1$ and $g_{22} = g_{\theta\theta} = r^2$ so that

$$R_{1212} = g_{1a} R^a{}_{212} = g_{11}\left(\partial_1 \Gamma^1_{22} - \partial_2 \Gamma^1_{21} + \Gamma^1_{b1}\Gamma^b_{22} - \Gamma^1_{b2}\Gamma^b_{21}\right)$$

From part (b), we have the only nonvanishing elements being $\Gamma^r_{\theta\theta} = -r$ and $\Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} = r^{-1}$:

$$R_{1212} = \partial_r \Gamma^r_{\theta\theta} - \Gamma^r_{\theta\theta}\Gamma^\theta_{\theta r} = -1 + r/r = 0.$$

13.3 **Symmetry property of Christoffel symbols** Because a scalar field $\Phi(x)$ is coordinate-independent, there is no difference between their covariant and ordinary derivatives, $D_\mu \Phi = \partial_\mu \Phi$. We then apply the result of Problem 13.1 to the torsion-free statement, after using $\partial_\nu \partial_\mu \Phi = \partial_\mu \partial_\nu \Phi$, to obtain

$$\begin{aligned} D_\nu D_\mu \Phi - D_\mu D_\nu \Phi &= -\Gamma^\lambda_{\nu\mu}\partial_\lambda \Phi + \Gamma^\lambda_{\mu\nu}\partial_\lambda \Phi \\ &= \left(-\Gamma^\lambda_{\nu\mu} + \Gamma^\lambda_{\mu\nu}\right)\partial_\lambda \Phi = 0. \end{aligned}$$

13.4 **Metric is covariantly constant: by explicit calculation** Take the covariant derivative of the metric tensor (with covariant indices) and then express the resulting Christoffel symbols in terms of derivatives of the metric

$$\begin{aligned} D_\mu g_{\nu\lambda} &= \partial_\mu g_{\nu\lambda} - \Gamma^\rho_{\mu\nu} g_{\rho\lambda} - \Gamma^\rho_{\mu\lambda} g_{\rho\nu} \\ &= \partial_\mu g_{\nu\lambda} - \frac{1}{2} g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) g_{\rho\lambda} \\ &\quad - \frac{1}{2} g^{\rho\sigma}\left(\partial_\mu g_{\lambda\sigma} + \partial_\lambda g_{\mu\sigma} - \partial_\sigma g_{\mu\lambda}\right) g_{\rho\nu}. \end{aligned}$$

After summing over repeated indices, we find all terms cancel.

13.5 **$D_\nu V_\mu$ is a good tensor: another proof** Start with (13.50) and use the fact that the σ -dependence is always through $x^\mu(\sigma)$

$$\frac{D}{d\sigma}\left(V_\mu \frac{dx^\mu}{d\sigma}\right) = \frac{D}{dx^\nu}\left(V_\mu \frac{dx^\mu}{d\sigma}\right)\frac{dx^\nu}{d\sigma} = 0.$$

We can use the geodesic equation in the form of $D(dx^\mu/d\sigma)/D\sigma = 0$ to obtain

$$\left(D_\nu V_\mu\right)\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma} = 0.$$

The quotient theorem then informs us that $D_\nu V_\mu$ is a good tensor, because it is contracted into a good tensor: $(dx^\mu/d\sigma)(dx^\nu/d\sigma)$.

13.6 **Parallel transport and the angular excess** The triangle has three vertices (A, B, C) connected by geodesic curves with interior angles (α, β, γ). We now transport a vector around this triangle, along the three geodesic sides of the triangle. The key observation is that the angle subtended by the vector and the geodesic is unchanged (cf. the worked example in the text).

1. At vertex A, the vector makes an angle θ_1 with the tangent along *AB*.
2. At vertex B, the vector makes the same angle θ_1 with the tangent along *AB,* thus it makes $\theta_2 = \theta_1 + (\pi - \beta)$ along *BC.*
3. At vertex C, the vector makes $\theta_3 = \theta_2 + (\pi - \gamma)$ along *CA.*
4. Returning to A, the vector makes $\theta_4 = \theta_3 + (\pi - \alpha)$ along the original *AB.*

Plug in θ_i sequentially and take out a trivial factor of 2π, we obtain the directional change of the vector

$$\delta\theta = \theta_1 - \theta_4 = \alpha + \beta + \gamma - \pi,$$

which is just the angular excess ϵ.

13.7 **Riemann curvature tensor as the commutator of covariant derivatives** Following the rule of (13.30), we have

$$\begin{aligned} D_\alpha D_\beta A_\mu &= \partial_\alpha \left(D_\beta A_\mu\right) - \underset{\text{drop}}{\underline{\Gamma^\nu_{\alpha\beta} D_\nu A_\mu}} - \Gamma^\nu_{\alpha\mu} D_\beta A_\nu \\ &= \underset{\text{drop}}{\underline{\partial_\alpha \partial_\beta A_\mu}} - \partial_\alpha \left(\Gamma^\nu_{\beta\mu} A_\nu\right) - \Gamma^\nu_{\alpha\mu}\partial_\beta A_\nu + \Gamma^\nu_{\alpha\mu}\Gamma^\lambda_{\beta\nu} A_\lambda \\ &= -\left(\partial_\alpha \Gamma^\lambda_{\beta\mu}\right) A_\lambda \underset{\text{drop}}{\underline{-\Gamma^\nu_{\beta\mu}\partial_\alpha A_\nu - \Gamma^\nu_{\alpha\mu}\partial_\beta A_\nu}} + \Gamma^\nu_{\alpha\mu}\Gamma^\lambda_{\beta\nu} A_\lambda. \end{aligned}$$

The underlined terms are symmetric in the indices (α, β) and will be cancelled when we include the $-D_\beta D_\alpha A_\mu$ calculation. From this we clearly get $D_\alpha, D_\beta A_\mu = -R^\lambda_{\mu\alpha\beta} A_\lambda$ with $R^\lambda_{\mu\alpha\beta}$ given by (13.58).

13.9 **Counting independent elements of Riemann tensor** Write the curvature tensor as $R_{\{\mu\nu,\alpha\beta\}}$ to remind ourselves the symmetry properties of (13.69) to (13.71): antisymmetry of Eq. (13.69) as $\mu\nu$, that of (13.70) as $\alpha\beta$, and the symmetry of (13.71) as $\{\mu\nu, \alpha\beta\}$. An $n \times n$ matrix has $\frac{1}{2}n(n+1)$ independent elements if it is symmetric, and $\frac{1}{2}n(n-1)$ if antisymmetric. Hence, for the purpose of counting independent components, we can regard $R_{\{\mu\nu,\alpha\beta\}}$ as a $\frac{1}{2}n(n-1)$ by $\frac{1}{2}n(n-1)$ matrix, which is symmetric. This yields a count of

$$\begin{aligned} M_{(n)} &= \frac{1}{2}\left[\frac{1}{2}n\,(n-1)\right] \times \left[\frac{1}{2}n\,(n-1)+1\right] \\ &= \frac{1}{8}n\,(n-1)\left(n^2 - n + 2\right). \end{aligned}$$

There are not as many independent elements as $M_{(n)}$ because we also need to factor-in the cyclic symmetry constraint of (13.72). Actually, (13.72) represents extra conditions that reduce the number of independent elements **only if** all four indices are different—because otherwise this cyclic condition reduces to the first three symmetry conditions. The number of additional constraint conditions as represented by (13.72) is given by:

$$C_{(n)} = \binom{n}{4} = n\,(n-1)\,(n-2)\,(n-3)\,/4$$

Subtracting $C_{(n)}$ from $M_{(n)}$ leads to the the number of independent components of a curvature tensor in an n -dimensional space:

$$N_{(n)} = M_{(n)} - C_{(n)} = \frac{1}{12}n^2(n^2-1). \tag{31}$$

13.10 **Counting metric's independent second derivatives**

(a) Remembering that the number of independent elements of a symmetric $n \times n$ matrix is $n(n+1)/2$, we see that the tensor $g_{\mu\nu}$ has 10 elements, and its first derivative $\partial_\alpha g_{\mu\nu}$ has 40, and its second derivative $\partial_\alpha\partial_\beta g_{\mu\nu}$

has 100 elements, when we used the fact that $\partial_\alpha \partial_\beta = \partial_\beta \partial_\alpha$. Namely,

	index sym	$A_{(4)}$
$g_{\mu\nu}$	$\{\mu\nu\}$	$(4 \times 5)/2 = 10$
$\partial_\alpha g_{\mu\nu}$	$\alpha\{\mu\nu\}$	$4 \times 10 = 40$
$\partial_\alpha \partial_\beta g_{\mu\nu}$	$\{\alpha\beta\}\{\mu\nu\}$	$10 \times 10 = 100$

In particular the number of components for the second derivative $\partial_\alpha \partial_\beta g_{\mu\nu}$ in an n-dimensional space is

$$A_{(n)} = \left[\frac{1}{2} n (n+1)\right]^2. \tag{32}$$

(b) Using the same notation as in (a), we find the number of parameters in the transformations for the four-dimensional space:

	index sym	$B_{(4)}$
$(\partial_\alpha x_\beta)$	$\alpha\beta$	$4 \times 4 = 16$
$\partial_\gamma (\partial_\alpha x_\beta)$	$\{\gamma\alpha\}\beta$	$10 \times 4 = 40$
$\partial_\gamma \partial_\delta (\partial_\alpha x_\beta)$	$\{\alpha\gamma\delta\}\beta$	$20 \times 4 = 80$

where, on the last line for the second derivative $\partial_\gamma \partial_\delta (\partial_\alpha x_\beta)$, we have used the fact that there are 20 possible totally symmetric combinations of three indices ($d = 3$) when each index can take on four possible values ($n = 4$). This is an example of the general result $N(d, n)$ being the number of symmetric combinations of d objects each can take on n possible values:

$$N(d, n) = \binom{d+n-1}{d} = \frac{(n+d-1)}{d\,(n-1)}. \tag{33}$$

One can understand this result by thinking of the ways, for example, of placing d identical balls into n boxes, which is equivalent to the problem of permuting d identical balls together with the $n-1$ partitions between the boxes.

(c) Of the results obtained in (a) and (b)

	$A_{(4)}$		$B_{(4)}$
$g_{\mu\nu}$	10	$(\partial_\alpha x_\beta)$	16
$\partial_\alpha g_{\mu\nu}$	40	$\partial_\gamma (\partial_\alpha x_\beta)$	40
$\partial_\alpha \partial_\beta g_{\mu\nu}$	100	$\partial_\gamma \partial_\delta (\partial_\alpha x_\beta)$	80

we note several features:

(i) **The $g_{\mu\nu}$ case** Do we need the 16 parameters of $(\partial_\alpha x_\beta)$ to determine the 10 elements of $g_{\mu\nu}$? Yes, this count is correct, because the transformation includes the six parameter Lorentz transformations that leave the Euclidean metric $g_{\mu\nu} = \eta_{\mu\nu}$ invariant.

(ii) **The $\partial_\alpha g_{\mu\nu}$ case** There are just the correct number (40) of parameters in $\partial_\gamma (\partial_\alpha x_\beta)$ to set all the 40 independent elements of $\partial_\alpha g_{\mu\nu}$ to zero. (Compare this to the flatness theorem.)

(iii) **The $\partial_\alpha \partial_\beta g_{\mu\nu}$ case** We still have 20 yet undetermined elements in the second derivative $\partial_\alpha \partial_\beta g_{\mu\nu}$. This just corresponds to the number of independent elements in the four dimensional curvature tensor $N_{(4)} = 20$ as shown in Problem 13.9.

(d) For a general n dimensional space, the number of second derivatives of the transformation $\partial_\gamma \partial_\delta \partial_\alpha x_\beta$ as given by (33) for $d = 3$ (with a further multiplication of n for the β index) is

$$B_{(n)} = \frac{1}{6} n^2 (n+2)(n+1). \tag{34}$$

The number of independent elements of the second derivative must be the difference of (32) and (34): $N_{(n)} = A_{(n)} - B_{(n)} = n^2(n^2-1)/12$, which exactly matches the result of (31).

13.11 **Reducing Riemann tensor to Gaussian curvature**

(a) For a two-dimensional space with orthogonal coordinates, we have the metrics

$$g_{\mu\nu} = \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} g^{11} & 0 \\ 0 & g^{22} \end{pmatrix}$$

with $g^{11} = 1/g_{11}$ and $g^{22} = 1/g_{22}$ so that $g_{\mu\nu} g^{\nu\lambda} = \delta^\lambda_\mu$. The Christoffel symbols can be calculated from

$$\Gamma^1_{\mu\nu} = \frac{1}{2} g^{11} \left(\partial_\mu g_{1\nu} + \partial_\nu g_{1\mu} - \partial_1 g_{\mu\nu} \right)$$

so that

$$\Gamma^1_{11} = \frac{1}{2g_{11}} \partial_1 g_{11}, \quad \Gamma^1_{22} = -\frac{1}{2g_{11}} \partial_1 g_{22}$$

$$\Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{2g_{11}} \partial_2 g_{11}.$$

Similarly, we also have

$$\Gamma^2_{22} = \frac{1}{2g_{221}} \partial_2 g_{22}, \qquad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2g_{22}} \partial_1 g_{22}.$$

The only nontrivial (and independent) curvature element is

$$\begin{aligned}
R_{1212} &= g_{1\mu} R^\mu_{212} \\
&= g_{11} \left(\partial_2 \Gamma^1_{21} - \partial_1 \Gamma^1_{22} + \Gamma^\nu_{21} \Gamma^1_{\nu 2} - \Gamma^\nu_{22} \Gamma^1_{\nu 1} \right) \\
&= g_{11} \left(\partial_2 \Gamma^1_{21} - \partial_1 \Gamma^1_{22} + \Gamma^1_{21} \Gamma^1_{12} + \Gamma^2_{21} \Gamma^1_{22} - \Gamma^1_{22} \Gamma^1_{1\nu 1} - \Gamma^2_{22} \Gamma^1_{21} \right) \\
&= \frac{1}{2} \left\{ \partial_2^2 g_{11} + \partial_1^2 g_{22} - \frac{1}{2g_{11}} \left[(\partial_1 g_{11})(\partial_1 g_{22}) + (\partial_2 g_{11})^2 \right] \right. \\
&\quad \left. - \frac{1}{2g_{22}} \left[(\partial_2 g_{11})(\partial_2 g_{22}) + (\partial_1 g_{22})^2 \right] \right\}
\end{aligned}$$

which, when divided by the metric determinant $g = g_{11} g_{22}$, the ratio $-R_{1212}/g$ is recognized as the Gaussian curvature of (5.35).

(b) The Ricci scalar is simply the twice contracted Riemann tensor $R = g^{\alpha\beta} g^{\mu\nu} R_{\alpha\mu\beta\nu} = 2g^{11} g^{22} R_{1212}$ because $R_{1212} = R_{2121}$. Since $g^{11} g^{22} = 1/g$, the result (a) leads to $R = -2K$.

(c) Equation (13.57) may be written as $dA^2 = R^2_{\ 112} A^1 \sigma$. Since the angular excess is related to the vector component change as $\epsilon = dA^2/A^1$, we can write this as

$$\epsilon = R^2_{\ 112}\sigma = -g^{22}R_{1212}\sigma = g^{22}gK\sigma$$
$$= g^{22}(g_{11}g_{22})K\sigma = K\sigma,$$

where we have used $g^{22}g_{22} = 1$ and $g_{11} = 1$ as, e.g. in polar system (r, θ).

13.12 **Bianchi identities** (1) The structure of the Bianchi identity (13.77) suggests that we consider the combination of double commutator of covariant derivatives, which manifestly vanishes (i.e. the Jacobi identity) when we expand out all the commutators:

$$\begin{aligned}
&\left[D_\lambda, \left[D_\mu, D_\nu\right]\right] + \left[D_\nu, \left[D_\lambda, D_\mu\right]\right] + \left[D_\mu, [D_\nu, D_\lambda]\right] \qquad (35)\\
&= D_\lambda D_\mu D_\nu - D_\lambda D_\nu D_\mu - D_\mu D_\nu D_\lambda + D_\nu D_\mu D_\lambda\\
&\quad + D_\nu D_\lambda D_\mu - D_\nu D_\mu D_\lambda - D_\lambda D_\mu D_\nu + D_\mu D_\lambda D_\nu\\
&\quad + D_\mu D_\nu D_\lambda - D_\mu D_\lambda D_\nu - D_\nu D_\lambda D_\mu + D_\lambda D_\nu D_\mu\\
&= 0.
\end{aligned}$$

(2) We now express these double commutators as covariant derivatives of the Riemann curvature tensors. We will find that D_λ, D_μ, D_ν is essentially $D_\lambda R_{\mu\nu\alpha\beta}$ with the extra terms from the three double commutators mutually cancel. Using the expression of the Riemann tensor in terms of commutator of covariant derivatives as worked out in Problem 13.7 (because $D_\lambda A_\alpha$ is a good tensor), we have

$$\begin{aligned}
\left[D_\lambda, \left[D_\mu, D_\nu\right]\right] A_\alpha &= D_\lambda \left[D_\mu, D_\nu\right] A_\alpha - \left[D_\mu, D_\nu\right] D_\lambda A_\alpha\\
&= -D_\lambda \left(R^\gamma_{\ \alpha\mu\nu} A_\gamma\right) + R^\gamma_{\ \alpha\mu\nu} D_\lambda A_\gamma + R^\gamma_{\ \lambda\mu\nu} D_\gamma A_\alpha\\
&= -D_\lambda R^\gamma_{\ \alpha\mu\nu} A_\gamma - R^\gamma_{\ \alpha\mu\nu} D_\lambda A_\gamma + R^\gamma_{\ \alpha\mu\nu} D_\lambda A_\gamma\\
&\quad + R^\gamma_{\ \lambda\mu\nu} D_\gamma A_\alpha\\
&= -D_\lambda R^\gamma_{\ \alpha\mu\nu} A_\gamma + R^\gamma_{\ \lambda\mu\nu} D_\gamma A_\alpha, \qquad (36)
\end{aligned}$$

as the two middle terms on the third line cancel. Applying this result to every double commutator, equation (35) can then be written as

$$\begin{aligned}
0 &= \left(\left[D_\lambda, \left[D_\mu, D_\nu\right]\right] + \left[D_\nu, \left[D_\lambda, D_\mu\right]\right] + \left[D_\mu, [D_\nu, D_\lambda]\right]\right) A_\alpha\\
&= -D_\lambda R^\gamma_{\ \alpha\mu\nu} A_\gamma - D_\nu R^\gamma_{\ \alpha\lambda\mu} A_\gamma - D_\mu R^\gamma_{\ \alpha\nu\lambda} A_\gamma\\
&\quad + R^\gamma_{\ \lambda\mu\nu} D_\gamma A_\alpha + R^\gamma_{\ \nu\lambda\mu} D_\gamma A_\alpha + R^\gamma_{\ \mu\nu\lambda} D_\gamma A_\alpha\\
&= -\left(D_\lambda R^\gamma_{\ \alpha\mu\nu} - D_\nu R^\gamma_{\ \alpha\lambda\mu} - D_\mu R^\gamma_{\ \alpha\nu\lambda}\right) A_\gamma\\
&\quad + \left(R^\gamma_{\ \lambda\mu\nu} + R^\gamma_{\ \nu\lambda\mu} + R^\gamma_{\ \mu\nu\lambda}\right) D_\gamma A_\alpha.
\end{aligned}$$

The second term on the RHS vanishes because of the cyclic symmetry property of (13.72); the parentheses in the prior term must then vanish, leading to the Bianchi identities.

13.14 **Contraction of Christoffel symbols** The inverse matrix ${g_{\mu\nu}}^{-1}$ has elements $g^{\mu\nu}$, which are related to the determinant g of the matrix $g_{\mu\nu}$ and the cofactors $C^{\mu\nu}$ (associated with elements $g_{\mu\nu}$) as

$$g^{\mu\nu} = \frac{C^{\mu\nu}}{g}. \tag{37}$$

Also, the determinant g can be expanded as (for any fixed μ)

$$g = \sum_\nu g_{\mu\nu} C^{\mu\nu} \tag{38}$$

where we have displayed the summation sign to emphasize that there is no summation over the index μ. Because the determinant is a function of the matrix elements $g_{\mu\nu}$ which in turn are position dependent, we have

$$\frac{\partial g}{\partial x^\alpha} = \frac{\partial g}{g_{\mu\nu}} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} = C^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} = g g^{\mu\nu} \partial_\alpha g_{\mu\nu} \tag{39}$$

where we have used (38) and (37) to reach the last two expressions. Knowing this identify, we proceed to make contraction of the Christoffel symbols

$$\Gamma^\mu_{\mu\alpha} = \frac{1}{2} g^{\mu\nu} \left[\partial_\alpha g_{\mu\nu} + \partial_\mu g_{\alpha\nu} - \partial_\nu g_{\mu\alpha} \right].$$

The last two terms cancel, $\partial^\nu g_{\alpha\nu} = \partial^\mu g_{\mu\alpha}$, so that the contraction can be rewritten by (39) as

$$\Gamma^\mu_{\mu\alpha} = \frac{1}{2} g^{\mu\nu} \partial_\alpha g_{\mu\nu} = \frac{1}{2g} \frac{\partial g}{\partial x^\alpha}$$

which is equivalent to the sought after result of

$$\Gamma^\mu_{\mu\alpha} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} \sqrt{-g}.$$

13.15 **Contraction of Riemann tensor** Contracting the first two indices $R^\mu_{\mu\alpha\beta}$ (13.58): $\partial_\alpha \Gamma^\mu_{\mu\beta} - \partial_\beta \Gamma^\mu_{\mu\alpha} + \Gamma^\mu_{\nu\alpha} \Gamma^\nu_{\mu\beta} - \Gamma^\mu_{\nu\beta} \Gamma^\nu_{\mu\alpha}$. The dummy indices in the last two terms can be relabelled $\mu \leftrightarrow \nu$; we see that they cancel each other. A straightforward calculation of the first two terms by using the result obtained in Problem 13.14 shows that they cancel each other also.

14.4 **The equation of geodesic deviation** Following the procedure used in Box 6.3, let us consider two particles: one has the spacetime trajectory x^μ and another has $x^\mu + s^\mu$. These two particles, separated by the displacement vector s^μ, obey the respective equations of motion:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}(x) \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0$$

and

$$\left(\frac{d^2 x^\mu}{d\tau^2} + \frac{d^2 s^\mu}{d\tau^2} \right) + \Gamma^\mu_{\alpha\beta}(x+s) \left(\frac{dx^\alpha}{d\tau} + \frac{ds^\alpha}{d\tau} \right) \left(\frac{dx^\beta}{d\tau} + \frac{ds^\beta}{d\tau} \right) = 0.$$

When the separation distance s^μ is small, we can approximate the Christoffel symbols $\Gamma^\mu_{\alpha\beta}(x+s)$ by a Taylor expansion

$$\Gamma^\mu_{\alpha\beta}(x+s) = \Gamma^\mu_{\alpha\beta}(x) + \partial_\lambda\Gamma^\mu_{\alpha\beta}s^\lambda + \cdots.$$

From the difference of the two geodesic equations, we obtain, to first order in s^μ,

$$\frac{d^2s^\mu}{d\tau^2} = -2\Gamma^\mu_{\alpha\beta}\frac{ds^\alpha}{d\tau}\frac{dx^\beta}{d\tau} - \partial_\lambda\Gamma^\mu_{\alpha\beta}s^\lambda\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}. \tag{40}$$

What we are seeking is the relative acceleration (the second derivative of the separation s^μ) along the worldline; thus, a double differentiation along the geodesic curve. From (13.47) we have the first derivative

$$\frac{Ds^\mu}{d\tau} = \frac{ds^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta}s^\alpha\frac{dx^\beta}{d\tau}$$

and the second derivative

$$\begin{aligned}\frac{D^2s^\mu}{d\tau^2} &= \frac{D}{d\tau}\left(\frac{Ds^\mu}{d\tau}\right) = \frac{d}{d\tau}\left(\frac{Ds^\mu}{d\tau}\right) + \Gamma^\mu_{\alpha\beta}\left(\frac{Ds^\alpha}{d\tau}\right)\frac{dx^\beta}{d\tau}\\ &= \frac{d}{d\tau}\left(\frac{ds^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta}s^\alpha\frac{dx^\beta}{d\tau}\right) + \Gamma^\mu_{\alpha\beta}\left(\frac{ds^\alpha}{d\tau} + \Gamma^\alpha_{\lambda\rho}s^\lambda\frac{dx^\rho}{d\tau}\right)\frac{dx^\beta}{d\tau}\\ &= \frac{d^2s^\mu}{d\tau^2} + \partial_\lambda\Gamma^\mu_{\alpha\beta}\frac{dx^\lambda}{d\tau}s^\alpha\frac{dx^\beta}{d\tau} + \Gamma^\mu_{\alpha\beta}\frac{ds^\alpha}{d\tau}\frac{dx^\beta}{d\tau} + \Gamma^\mu_{\alpha\beta}s^\alpha\frac{d^2x^\beta}{d\tau^2}\\ &\quad + \Gamma^\mu_{\alpha\beta}\frac{ds^\alpha}{d\tau}\frac{dx^\beta}{d\tau} + \Gamma^\mu_{\alpha\beta}\Gamma^\alpha_{\lambda\rho}s^\lambda\frac{dx^\rho}{d\tau}\frac{dx^\beta}{d\tau}.\end{aligned} \tag{41}$$

For the $d^2s^\mu/d\tau^2$ term we use (40); for the $d^2x^\beta/d\tau^2$ term we use the geodesic equation

$$\frac{d^2x^\beta}{d\tau^2} = -\Gamma^\beta_{\lambda\rho}\frac{dx^\lambda}{d\tau}\frac{dx^\rho}{d\tau}.$$

This way one finds

$$\begin{aligned}\frac{D^2s^\mu}{d\tau^2} = &-2\Gamma^\mu_{\alpha\beta}\frac{ds^\alpha}{d\tau}\frac{dx^\beta}{d\tau} - \partial_\lambda\Gamma^\mu_{\alpha\beta}s^\lambda\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} + \partial_\lambda\Gamma^\mu_{\alpha\beta}\frac{dx^\lambda}{d\tau}s^\alpha\frac{dx^\beta}{d\tau}\\ &+ 2\Gamma^\mu_{\alpha\beta}\frac{ds^\alpha}{d\tau}\frac{dx^\beta}{d\tau} - \Gamma^\mu_{\alpha\beta}s^\alpha\Gamma^\beta_{\lambda\rho}\frac{dx^\lambda}{d\tau}\frac{dx^\rho}{d\tau}\\ &+ \Gamma^\mu_{\alpha\beta}\Gamma^\alpha_{\lambda\rho}s^\lambda\frac{dx^\rho}{d\tau}\frac{dx^\beta}{d\tau}.\end{aligned}$$

After a cancellation of two terms and relabeling of several dummy indices, this becomes

$$\begin{aligned}\frac{D^2s^\mu}{d\tau^2} = &-\partial_\lambda\Gamma^\mu_{\alpha\beta}s^\lambda\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} + \partial_\alpha\Gamma^\mu_{\lambda\beta}\frac{dx^\alpha}{d\tau}s^\lambda\frac{dx^\beta}{d\tau}\\ &-\Gamma^\mu_{\lambda\rho}s^\lambda\Gamma^\rho_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} + \Gamma^\mu_{\rho\beta}\Gamma^\rho_{\lambda\alpha}s^\lambda\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\end{aligned}$$

or

$$\frac{D^2 s^\mu}{d\tau^2} = -R^\mu_{\alpha\lambda\beta} s^\lambda \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau},$$

where

$$R^\mu_{\alpha\lambda\beta} = \partial_\lambda \Gamma^\mu_{\alpha\beta} - \partial_\beta \Gamma^\mu_{\lambda\alpha} + \Gamma^\mu_{\lambda\rho}\Gamma^\rho_{\alpha\beta} - \Gamma^\mu_{\beta\rho}\Gamma^\rho_{\lambda\alpha}$$

in agreement with (13.58).

14.5 **From geodesic deviation to NR tidal forces** Besides slow moving particles, the Newtonian limit means a weak gravitational field: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $h_{\mu\nu}$ being small. Thus (13.37) becomes

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}\eta^{\mu\rho}\partial_\alpha h_{\beta\rho} + \partial_\beta h_{\alpha\rho} - \partial_\rho h_{\alpha\beta}.$$

Also, in this weak-field limit, we can drop the quadratic terms $(\Gamma\Gamma)$ in the curvature so that there are only two terms, related by the interchange of (β, λ) indices

$$\begin{aligned} R^\mu_{\alpha\lambda\beta} &= \partial_\lambda \Gamma^\mu_{\alpha\beta} - \partial_\beta \Gamma^\mu_{\lambda\alpha} \\ &= \frac{1}{2}\eta^{\mu\rho}\partial_\lambda\partial_\alpha h_{\beta\rho} - \partial_\lambda\partial_\rho h_{\alpha\beta} - \partial_\beta\partial_\alpha h_{\lambda\rho} + \partial_\beta\partial_\rho h_{\alpha\lambda} \end{aligned}$$

after cancelling two terms. Thus

$$R^i_{0j0} = \frac{1}{2}\partial_j\partial_0 h_{0i} - \partial_j\partial_i h_{00} - \partial_0\partial_0 h_{ji} + \partial_0\partial_i h_{0j} = -\frac{1}{2}\partial_i\partial_j h_{00}.$$

Because the Newtonian limit also has the static field condition, to reach the last line we have dropped all time derivatives. With $h_{00} = -2\Phi/c^2$ as given by (6.20), we have the sought-after relation of

$$R^i_{0j0} = \frac{1}{c^2}\frac{\partial^2\Phi}{\partial x^i \partial x^j}.$$

14.6 **Relativistic spin precession** The Schwarzschild metric for a circular orbit (radius R) in the equatorial plane $(\theta = \pi/2)$ has elements of

$$g_{tt} = -\left(1 - \frac{r^*}{R}\right), \quad g_{rr} = \left(1 - \frac{r^*}{R}\right)^{-1}, \quad g_{\theta\theta} = g_{\phi\phi} = R^2. \tag{42}$$

From this and the orthogonality condition $S^\mu U_\mu = 0$ we can immediately deduce the proportionality relation between S^t and S^ϕ:

$$g_{\alpha\beta} S^\alpha U^\beta = -\left(1 - \frac{r^*}{R}\right) S^t U^t + R^2 S^\phi \Omega U^t = 0$$

or

$$S^t = \left(1 - \frac{r^*}{R}\right)^{-1} R^2 \Omega S^\phi. \tag{43}$$

From (42) we can calculate the Christoffel symbols, which mostly vanish, with the nonzero ones being

$$\Gamma^r_{rt} = \frac{r^*}{2R^2}\left(1-\frac{r^*}{R}\right), \quad \Gamma^r_{\phi\phi} = -R\left(1-\frac{r^*}{R}\right),$$
$$\Gamma^t_{rt} = \frac{r^*}{2R^2}\left(1-\frac{r^*}{R}\right)^{-1}, \quad \Gamma^\phi_{r\phi} = \frac{1}{R}. \tag{44}$$

The gyroscope equation $DS^\mu/d\tau = 0$, for the $\mu = \phi$ component, may then be written out as

$$\frac{dS^\phi}{d\tau} + \Gamma^\phi_{r\phi} S^r U^\phi = 0.$$

Substituting in the Christoffel symbol value and replacing the proper time derivative by the coordinate time derivative $d/d\tau = U^t d/dt$, we get

$$\frac{dS^\phi}{dt} + \frac{\Omega}{R} S^r = 0. \tag{45}$$

For the $\mu = r$ component, we have

$$\frac{dS^r}{d\tau} + \Gamma^r_{tt} S^t U^t + \Gamma^r_{\phi\phi} S^\phi U^\phi = 0$$

which, using the relation (43), becomes

$$\frac{dS^r}{dt} - R\Omega\left(1-\frac{3r^*}{R}\right) S^\phi = 0. \tag{46}$$

For the two other components, $dS^\theta/d\tau = dS^\theta/dt = 0$ leads to $S^\theta(t) = S^\theta(0) = 0$, while the $\mu = t$ equation can be shown to be identical to (45). We can now solve for $S^r(t)$ by first time-differentiating Eq. (46)

$$\frac{d^2S^r}{dt^2} - R\Omega\left(1-\frac{3r^*}{R}\right)\frac{dS^\phi}{dt} = 0$$

and plug in the expression for dS^ϕ/dt from (45) to obtain

$$\frac{d^2S^r}{dt^2} + \Omega'^2 S^r = 0 \tag{47}$$

with

$$\Omega' = \left(1-\frac{3r^*}{R}\right)^{1/2}\Omega. \tag{48}$$

The simple harmonic oscillator equation (47), with the initial condition of (14.77), has the standard solution

$$S^r(t) = S^r_0 \cos\Omega' t.$$

The S^ϕ component can then be gotten by (46)

$$S^\phi = \frac{\Omega}{R\Omega'^2}\frac{dS^r}{dt} = -\frac{\Omega}{R\Omega'} S^r_0 \sin\Omega' t.$$

15.1 Gauge transformations

(a) Consider a coordinate (gauge) transformation as given in (15.12) so that, according to (15.17), $h'_{\alpha\beta} = h_{\alpha\beta} - \partial_\alpha \chi_\beta - \partial_\beta \chi_\alpha$. This implies (by contracting the indices on both sides) the transformation for the trace $h' = h - 2\partial^\beta \chi_\beta$. These two relations can be combined to yield the gauge transformation of $\bar{h}_{\alpha\beta}$,

$$h'_{\alpha\beta} - \frac{h'}{2}\eta_{\alpha\beta} = \bar{h}'_{\alpha\beta} = \bar{h}_{\alpha\beta} - \partial_\alpha \chi_\beta - \partial_\beta \chi_\alpha + \eta_{\alpha\beta}(\partial^\gamma \chi_\gamma). \quad (49)$$

(b) Taking the derivative on both sides of (49), $\partial^\alpha \bar{h}'_{\alpha\beta} = \partial^\alpha \bar{h}_{\alpha\beta} - \Box \chi_\beta$. The new metric perturbation field can be made to obey the Lorentz condition $\partial^\alpha \bar{h}'_{\alpha\beta} = 0$, if $\Box \chi_\beta = \partial^\alpha \bar{h}_{\alpha\beta}$.

(c) Plugging $\bar{h}_{\mu\nu} = \epsilon_{\mu\nu} e^{ikx}$ and $\chi_\nu = X_\nu e^{ikx}$ into the gauge transformation (49), we have

$$\epsilon'_{\mu\nu} = \epsilon_{\mu\nu} - ik_\mu X_\nu - ik_\nu X_\mu + i\eta_{\mu\nu}(k \cdot X) \quad (50)$$

which implies the trace relation $\epsilon'^\mu_\mu = \epsilon^\mu_\mu + 2ik^\mu X_\mu$. This means that if we start with a polarization tensor that is not traceless, it will be traceless $\epsilon'^\mu_\mu = 0$ in a new coordinate if the gauge vector function X_μ for the coordinate transformation is chosen to satisfy the condition $2ik \cdot X = -\epsilon^\mu_\mu$. Now we have used one of the four numbers in X_μ to fix the trace. How can we use the remaining three to obtain $\epsilon_{\mu 0} = 0$ which would seem to represent four conditions? This is possible because we are working in the Lorentz gauge and k^μ is a null-vector. Here is the reason. Starting with $\epsilon_{\mu 0} \neq 0$, new coordinate transformation leads to (50) with

$$\epsilon'_{\mu 0} = \epsilon_{\mu 0} - ik_\mu X_0 - ik_0 X_\mu + i\eta_{\mu 0}(k \cdot X).$$

Formally $\epsilon'_{\mu 0} = 0$ represents four conditions. But, because of $k^\mu \epsilon_{\mu 0} = 0$ and $k^2 = 0$, these four equations must obey a constraint relation, obtained by a contraction with the vector k^μ:

$$k^\mu \epsilon_{\mu 0} - ik^2 X_0 - ik_0(k \cdot X) + ik_0(k \cdot X) = 0.$$

That is, $k^\mu \epsilon'_{\mu 0} = 0$. Thus, $\epsilon'_{\mu 0} = 0$ actually stands for three independent relations.

(d) The polarization tensor being symmetric, $\epsilon_{\mu\nu} = \epsilon_{\nu\mu}$, it has 10 independent elements. The Lorentz gauge condition $k^\mu \epsilon_{\mu\nu} = 0$ represents four constraints, $\epsilon^\mu_\mu = 0$ is one, and $\epsilon_{\mu 0} = 0$, as discussed above, is three. Thus there are only $10 - 4 - 1 - 3 = 2$ independent elements in the polarization tensor.

15.2 The Schwarzschild solution We work first in the Cartesian coordinates $x^\mu = (ct, x^i)$ with an approximate flat metric according to (15.1). The solution (15.74) must fulfill the Lorentz gauge condition (15.18)

$$\partial^\mu \bar{h}_{\mu\nu} = C_{\mu\nu}\partial^\mu \left(\frac{1}{r}\right) = C_{i\nu}\partial^i \left(\frac{1}{r}\right) = -\frac{x^i C_{i\nu}}{r} = 0.$$

for any x^i. This can only be satisfied by $C_{i\nu} = 0$. That is, every element of $\bar{h}_{\mu\nu}$ vanishes except $\bar{h}_{00} = C_{00}/r$. This also means that the trace $\bar{h} = \eta^{\mu\nu}\bar{h}_{\mu\nu} = -C_{00}/r$ and $h = \eta^{\mu\nu}h_{\mu\nu} = C_{00}/r$. From (15.19) we find the perturbation element $h_{00} = C_{00}/2r$. Because of spherical symmetry, we have $h_{11} = h_{22} = h_{33}$, and, to have the correct trace, each must equal to $C_{00}/2r$; namely, $h_{\mu\nu} = (C_{00}/2r)\delta_{\mu\nu}$. We can also fix the constant C_{00} by its Newtonian value as in (15.1) and (6.3)

$$g_{00} = -1 + \frac{C_{00}}{2r} = -1 - \frac{2\Phi}{c^2} = -1 + \frac{2G_{\rm N}M}{c^2 r},$$

or $C_{00} = 4G_{\rm N}M/c^2 = 2r^*$. In this way we find the approximate Schwarzschild metric in Cartesian coordinates as

$$ds^2 = -\left(1 - \frac{r^*}{r}\right)c^2dt^2 + \left(1 + \frac{r^*}{r}\right)\left(dx_1^2 + dx_2^2 + dx_3^2\right).$$

In terms of the spherical coordinate we have

$$\left[dx_1^2 + dx_2^2 + dx_3^2\right] = \left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right].$$

In order to show our result in a form closer to the exact Schwarzschild metric of (14.54), we make the coordinate change of $\bar{r}^2 = (1 + (r^*/r))r^2$ so that, within the approximation of dropping $O(r^{*2})$ terms, we have

$$\frac{r^*}{r} \simeq \frac{r^*}{\bar{r}} \quad \text{and} \quad dr = d\bar{r}$$

and

$$ds^2 = -\left(1 - \frac{r^*}{\bar{r}}\right)c^2dt^2 + \left(1 + \frac{r^*}{\bar{r}}\right)d\bar{r}^2 + \bar{r}^2\left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

This is the expected result of (15.75).

15.3 **Wave effect via the deviation equation** With a collection of nearby particles, we can consider velocity and separation fields, $U^\mu(x)$ and $S^\mu(x)$. The equation of geodesic deviation (Problem 14.4) may be written as

$$\frac{D^2}{d\tau^2}S^\mu = R^\mu_{\ \nu\lambda\rho}U^\nu U^\lambda S^\rho.$$

Since a slow moving particle $U^\mu = (c, 0, 0, 0) + O(h)$ and the Riemann tensor $R^\mu_{\ \nu\lambda\rho} = O(h)$, this equation has the structure

$$\frac{D^2}{d\tau^2}S^\mu = c^2\eta^{\mu\sigma}R^{(1)}_{\sigma 00\rho}S^\rho + O(h^2).$$

The Christoffel symbols being of higher order, the covariant derivative may be replaced by ordinary differentiation; this equation at $O(h)$ is

$$\frac{d^2S^\mu}{dt^2} = \frac{S^\rho}{2}\frac{d^2}{dt^2}h^\mu_\rho.$$

On the RHS we have used (15.6) and the TT gauge condition of $h_{00} = h_{0\mu} = 0$. The longitudinal component of the separation field S_z is not affected because $h_{3\rho} = 0$ in the TT gauge. For an incoming wave in the "plus" polarization state, the transverse components obey the equations

$$\frac{d^2 S_x}{dt^2} = \frac{S_x}{2}\frac{d^2}{dt^2}(h_+ e^{i(kx-\omega t)}), \quad \frac{d^2 S_y}{dt^2} = -\frac{S_y}{2}\frac{d^2}{dt^2}(h_+ e^{i(kx-\omega t)}).$$

These equations, to the lowest order, have solutions

$$S_x(x) = \left(1 + \frac{1}{2}h_+ e^{i(kx-\omega t)}\right) S_x(0),$$
$$S_y(x) = \left(1 - \frac{1}{2}h_+ e^{i(kx-\omega t)}\right) S_y(0)$$

in agreement with the result in (15.30) and (15.31).

15.4 $\Gamma^{\mu}_{\nu\lambda}$ **and** $R^{(2)}_{\mu\nu}$ **in the TT gauge**

(a) Christoffel symbols: we give samples of the calculation

$$\Gamma^1_{00} = \frac{1}{2}g^{11}(\partial_0 g_{10} + \partial_0 g_{01} - \partial_1 g_{00}) = 0$$

because $h_{10} = h_{01} = h_{00} = 0$ in the TT gauge:

$$\Gamma^1_{01} = \frac{1}{2}(1 - \tilde{h}_{11})(\partial_0 \tilde{h}_{11} + \partial_1 \tilde{h}_{01} - \partial_1 \tilde{h}_{10})$$
$$= \frac{1}{2}(\partial_0 \tilde{h}_+ - \tilde{h}_+ \partial_0 \tilde{h}_+).$$

(b) Ricci tensor: from what we know of Christoffel symbols having the non-vanishing elements of

$$\Gamma^1_{10} = \Gamma^1_{01} = \Gamma^0_{11} = \frac{1}{2}\partial_0 \tilde{h}_+,$$
$$\Gamma^1_{13} = \Gamma^1_{31} = -\Gamma^3_{11} = -\frac{1}{2}\partial_0 \tilde{h}_+$$

together with the same terms with the replacement of indices from 1 to 2, we can calculate the second-order Ricci tensor by

$$R^{(2)}_{\mu\nu} = \Gamma^{\alpha}_{\alpha\lambda}\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\alpha}_{\mu\lambda}\Gamma^{\lambda}_{\alpha\nu}.$$

Thus

$$R^{(2)}_{00} = \Gamma^{\alpha}_{\alpha\lambda}\Gamma^{\lambda}_{00} - \Gamma^{\alpha}_{0\lambda}\Gamma^{\lambda}_{\alpha 0}$$
$$= 0 - 2\Gamma^1_{01}\Gamma^1_{10} = \frac{-1}{2}(\partial_0 \tilde{h}_+)^2 = R^{(2)}_{33},$$
$$R^{(2)}_{11} = \Gamma^{\alpha}_{\alpha\lambda}\Gamma^{\lambda}_{11} - \Gamma^{\alpha}_{1\lambda}\Gamma^{\lambda}_{\alpha 1}$$
$$= 2\Gamma^1_{1\lambda}\Gamma^{\lambda}_{11} - \Gamma^0_{1\lambda}\Gamma^{\lambda}_{01} - \Gamma^1_{1\lambda}\Gamma^{\lambda}_{11} - \Gamma^3_{1\lambda}\Gamma^{\lambda}_{31}$$

$$= 2\Gamma^1_{10}\Gamma^0_{11} + 2\Gamma^1_{13}\Gamma^3_{11} - \Gamma^0_{11}\Gamma^1_{01} - \Gamma^1_{10}\Gamma^0_{11} - \Gamma^1_{13}\Gamma^3_{11} - \Gamma^3_{11}\Gamma^1_{31}$$
$$= 0 = R^{(2)}_{22}.$$

15.5 **Trace calculation of $\tilde{I}^{\mathrm{TT}}_{ij}$** From the definition (15.58), we have

$$\tilde{I}^{\mathrm{TT}}_{ij} = \Pi_{ik}\Pi_{jl}\tilde{I}_{kl} - \frac{1}{2}\Pi_{ij}\left(\Pi_{kl}\tilde{I}_{kl}\right).$$

To calculate its trace, we need to compute $\delta_{ij}\Pi_{ik}\Pi_{jl}$ and $\delta_{ij}\Pi_{ij}$:

$$\delta_{ij}\Pi_{ik}\Pi_{jl} = \delta_{ij}\left(\delta_{ik} - n_i n_k\right)\left(\delta_{jl} - n_j n_l\right)$$
$$= \left(\delta_{kl} - n_k n_l\right) = \Pi_{kl}.$$

Since $\delta_{ij}\Pi_{ij} = \delta_{ij}\left(\delta_{ij} - n_i n_j\right) = 3 - 1 = 2$, we have the trace of $\tilde{I}^{\mathrm{TT}}_{ij}$ as

$$\delta_{ij}\tilde{I}^{\mathrm{TT}}_{ij} = \left(\delta_{ij}\Pi_{ik}\Pi_{jl}\right)\tilde{I}_{kl} - \frac{1}{2}\left(\delta_{ij}\Pi_{ij}\right)\Pi_{kl}\tilde{I}_{kl}$$
$$= \Pi_{kl}\tilde{I}_{kl} - \Pi_{kl}\tilde{I}_{kl} = 0.$$

15.6 **Derive the relation (15.59)** From the definition of (15.58) and the shorthand $\left(\Pi\tilde{I}\right) = \Pi_{kl}\tilde{I}_{kl}$, we have

$$\tilde{I}^{\mathrm{TT}}_{ij}\tilde{I}^{\mathrm{TT}}_{ij} = \left[\Pi_{ik}\Pi_{jl}\tilde{I}_{kl} - \frac{1}{2}\Pi_{ij}\left(\Pi\tilde{I}\right)\right]\left[\Pi_{im}\Pi_{jn}\tilde{I}_{mn} - \frac{1}{2}\Pi_{ij}\left(\Pi\tilde{I}\right)\right].$$

Using the result, $\Pi_{ik}\Pi_{il} = \Pi_{kl}$, obtained in Problem 15.5 we can carrying out the various multiplications:

$$\Pi_{ik}\Pi_{jl}\Pi_{im}\Pi_{jn}\tilde{I}_{kl}\tilde{I}_{mn} = \left(\Pi_{ik}\Pi_{im}\right)\left(\Pi_{jl}\Pi_{jn}\right)\tilde{I}_{kl}\tilde{I}_{mn}$$
$$= \Pi_{km}\Pi_{nl}\tilde{I}_{kl}\tilde{I}_{mn}$$
$$= \left(\delta_{km} - n_k n_m\right)\left(\delta_{nl} - n_n n_l\right)\tilde{I}_{kl}\tilde{I}_{mn}$$
$$= \tilde{I}_{ij}\tilde{I}_{ij} - 2n_k n_l\tilde{I}_{ki}\tilde{I}_{li} + n_k n_l n_m n_n\tilde{I}_{kl}\tilde{I}_{mn}$$

and

$$\Pi_{ij}\Pi_{ij}\left(\Pi\tilde{I}\right)^2 = \left(\delta_{ij} - n_i n_j\right)\left(\delta_{ij} - n_i n_j\right)\left(\Pi\tilde{I}\right)^2 = 2\left(\Pi\tilde{I}\right)^2,$$

and

$$\Pi_{ij}\Pi_{im}\Pi_{jn}\tilde{I}_{mn}\left(\Pi\tilde{I}\right) = \Pi_{jm}\Pi_{jn}\tilde{I}_{mn}\left(\Pi\tilde{I}\right) = \Pi_{mn}\tilde{I}_{mn}\left(\Pi\tilde{I}\right) = \left(\Pi\tilde{I}\right)^2$$

so we have

$$\tilde{I}^{\mathrm{TT}}_{ij}\tilde{I}^{\mathrm{TT}}_{ij} = \tilde{I}_{ij}\tilde{I}_{ij} - 2n_k n_l\tilde{I}_{ki}\tilde{I}_{li} + n_k n_l n_m n_n\tilde{I}_{kl}\tilde{I}_{mn} - \frac{1}{2}\left(\Pi\tilde{I}\right)^2$$
$$= \frac{1}{2}\left[2\tilde{I}_{ij}\tilde{I}_{ij} - 4\tilde{I}_{ik}\tilde{I}_{il}n_k n_l + \tilde{I}_{ij}\tilde{I}_{kl}n_i n_j n_k n_l\right]$$

because $\left(\Pi\tilde{I}\right) = -n_i n_j\tilde{I}_{ij}$ and $\left(\Pi\tilde{I}\right)^2 = \tilde{I}_{ij}\tilde{I}_{kl}n_i n_j n_k n_l$.