Linearized theory and gravitational waves

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- In the weak-field limit Einstein's equation can be linearized and it takes on form of the familiar wave equation.
- Gravitational waves may be viewed as ripples of curvature propagating in a background of flat spacetime.
- The strategy of detecting such tidal forces by a gravitational wave interferometer is outlined.
- The rate of energy loss due to the quadrupole radiation by a circulating binary system is calculated, and found to be in excellent agreement with the observed orbit decay rate of the Hulse–Taylor binary pulsar.

Newton's theory of gravitation is a static theory. The Newtonian field due to a source is established instantaneously. Thus, while the field has nontrivial dependence on the spatial coordinates, it does not depend on time. Einstein's theory, being relativistic, treats space and time on an equal footing. Just like Maxwell's theory, it has the feature that a field propagates outward from the source with a finite speed. In this chapter we study the case of a weak gravitational field. This approximation linearizes the Einstein theory. In this limit, a gravitational waves may be viewed as small curvature ripples (the metric field) propagating in a background of flat spacetime. It is a transverse wave, having two independent polarization states, traveling at the speed of light.

Because gravitational interaction is so weak, any significant emission of gravitational radiation can come only from a strong field region involving dynamics that directly reflects GR physics. Once gravitational waves are emitted, they will not scatter and they propagate out undisturbed from the inner core of an imploding star, from the arena of black hole formation, and from the earliest moments of the universe, etc. That is, they come from regions which are usually obscured in electromagnetic, even neutrino astronomy: gravitational waves can provide us with a new window into astrophysical phenomena.

These ripples of curvature can be detected as tidal forces. We provide an outline of the detection strategy using gravitational wave interferometers, which can measure the minute compression and elongation of orthogonal lengths that are caused by the passage of such a wave. In the final section, we present the indirect, but convincing, evidence for the existence of gravitational waves as predicted by general relativity (GR). This came from the observation, spanning more than 25 years, of orbital motion of the relativistic Hulse–Taylor binary pulsar system (PSR 1913+16). Even though the binary pair is 5 kpc away from us, the basic parameters of the system can be deduced by carefully monitoring the radio pulses emitted by the pulsar, which effectively acted as an accurate and stable clock. From this record we can verify a number of GR effects. In particular the orbital period is observed to decrease. According to GR, this is brought about by the gravitational wave quadrupole radiation from the system. The observed orbital rate decrease is in splendid agreement with the prediction by Einstein's theory.

15.1 Linearized theory of a metric field

Even though the production of gravitational waves usually involves strong field situations, but, because of the weakness of the gravitational interaction, the produced gravitational waves are only tiny displacements of the flat spacetime metric. Thus it is entirely adequate for the description of a gravity wave to restrict ourselves to the situation of a weak gravitation field. In this limit,[1] the metric is almost Minkowskian $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \equiv g^{(1)}_{\mu\nu} \tag{15.1}$$

[1]Contrasting this to the Newtonian limit of nonrelativistic motion in a weak static field, here we remove the restriction of slow motion and allow for a time-dependent field.

where the metric perturbation $|h_{\mu\nu}| \ll 1$ everywhere in spacetime. Thus we will keep only first-order terms in $h_{\mu\nu}$, and denote the relevant quantities with a superscript $^{(1)}$. The idea is that slightly curved coordinate systems exist and they are suitable coordinates to use in the weak field situation. We can still make coordinate transformations among such systems—from one slightly curved one to another. In particular we can make a "background Lorentz transformation." Distinguishing the indices, $\{\mu\}$ vs. $\{\mu'\}$, to indicate the pre-transformed and transformed coordinates, we have

$$x^{\mu} \to x^{\mu'} = [\mathbf{L}]^{\mu'}_{\ \nu} x^{\nu} \tag{15.2}$$

where $\mathbf{L}$ is the position-independent Lorentz transformation of special relativity, see (12.13) and (12.17). The key property of such transformations is that they keep the Minkowski metric invariant, see (12.19),

$$\left[\mathbf{L}^{-1}\right]^{\mu}_{\ \alpha'} \left[\mathbf{L}^{-1}\right]^{\nu}_{\ \beta'} \eta_{\mu\nu} = \eta_{\alpha'\beta'} \tag{15.3}$$

This leads to the transformation of the full metric as

$$\left[\mathbf{L}^{-1}\right]^{\mu}_{\ \alpha'} \left[\mathbf{L}^{-1}\right]^{\nu}_{\ \beta'} g^{(1)}_{\mu\nu} = \eta_{\alpha'\beta'} + \left[\mathbf{L}^{-1}\right]^{\mu}_{\ \alpha'} \left[\mathbf{L}^{-1}\right]^{\nu}_{\ \beta'} h_{\mu\nu} = g^{(1)}_{\alpha'\beta'}. \tag{15.4}$$

Thus

$$h_{\alpha'\beta'} = \left[\mathbf{L}^{-1}\right]^{\mu}_{\ \alpha'} \left[\mathbf{L}^{-1}\right]^{\nu}_{\ \beta'} h_{\mu\nu}. \tag{15.5}$$

Namely, $h_{\mu\nu}$ is just a Lorentz tensor. Thus this part of the metric can be taken as a tensor defined on a flat Minkowski spacetime. Since the nontrivial physics is contained in $h_{\mu\nu}$, we can have the convenient picture of a weak gravitational field as being described by this symmetric field $h_{\mu\nu}$ in a flat spacetime.[2]

Dropping higher order terms of $h_{\mu\nu}$, we have the Riemann curvature tensor

$$R^{(1)}_{\alpha\mu\beta\nu} = \frac{1}{2}\left(\partial_\alpha\partial_\nu h_{\mu\beta} + \partial_\mu\partial_\beta h_{\alpha\nu} - \partial_\alpha\partial_\beta h_{\mu\nu} - \partial_\mu\partial_\nu h_{\alpha\beta}\right), \tag{15.6}$$

the Ricci tensor

$$\begin{aligned} R^{(1)}_{\mu\nu} &= \eta^{\alpha\beta} R^{(1)}_{\alpha\mu\beta\nu} \\ &= \frac{1}{2}\left(\partial_\alpha\partial_\nu h^{\alpha}_{\ \mu} + \partial_\mu\partial_\alpha h^{\alpha}_{\ \nu} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h\right), \end{aligned} \tag{15.7}$$

and the Ricci scalar

$$R^{(1)} = \partial_\mu\partial_\nu h^{\mu\nu} - \Box h, \tag{15.8}$$

where $\Box = \partial_\mu\partial^\mu$ and $h = h^\mu_\mu$ is the trace. Clearly the resultant Einstein tensor

$$G^{(1)}_{\mu\nu} = R^{(1)}_{\mu\nu} - \frac{1}{2}R^{(1)}\eta_{\mu\nu} \tag{15.9}$$

is also linear in $h_{\mu\nu}$, and so is the Einstein equation:

$$G^{(1)}_{\mu\nu} = -\frac{8\pi G_N}{c^4} T^{(0)}_{\mu\nu}. \tag{15.10}$$

For a spacetime being slightly curved the left hand side (LHS) is of order $h_{\mu\nu}$; this means that the energy–momentum tensor must also be small, $T^{(0)}_{\mu\nu} = O(h_{\mu\nu})$.Thus, its conservation condition $D^\mu T_{\mu\nu} = 0$ can be simply expressed in terms of ordinary derivatives

$$\partial^\mu T^{(0)}_{\mu\nu} = 0 \tag{15.11}$$

as the difference between D_μ and ∂_μ is of the order of $h_{\mu\nu}$.

15.1.1 The coordinate change called a gauge transformation

In the following, we shall make coordinate transformations so that the linearized Einstein equation (15.10) can be written more compactly in terms of $h_{\mu\nu}$. This class of coordinate transformations (within the slightly curved spacetime) is called, collectively, **gauge transformations** because of their close resemblance to the electromagnetic gauge transformations. Consider a small shift of the position vector:

$$x^{\mu'} = x^\mu + \chi^\mu(x) \tag{15.12}$$

where $\chi^\mu(x)$ are four arbitrary small functions. Collectively they are called the "vector gauge function" (as opposed to the scalar gauge function in electromagnetic gauge transformations see Box 12.3). Clearly this is not a tensor equation, as indices do not match on the two sides. (Our notation indicates the relation of the position vector as labeled by the transformed

[2] Eventually in a quantum description, $h_{\mu\nu}$ is a field for the spin-2 gravitons, and the perturbative description of gravitational interactions as due to the exchanges of massless gravitons.

and pre-transformed coordinates.) The transformation matrix elements (for the contravariant components) can be obtained by differentiating (15.12):

$$\frac{\partial x^{\mu'}}{\partial x^{\alpha}} = \delta^{\mu}_{\alpha} + \partial_{\alpha}\chi^{\mu}. \tag{15.13}$$

The smallness of the shift $\chi \ll x$ means

$$\left|\partial_{\mu}\chi^{\nu}\right| \ll 1. \tag{15.14}$$

This implies an inverse transformation of

$$\frac{\partial x^{\mu}}{\partial x^{\alpha'}} = \delta^{\mu}_{\alpha} - \partial_{\alpha}\chi^{\mu} + O\left(|\partial\chi|^2\right). \tag{15.15}$$

Apply it to the metric tensor:[3]

$$\begin{aligned} g^{(1)}_{\alpha'\beta'} &= \frac{\partial x^{\mu}}{\partial x^{\alpha'}}\frac{\partial x^{\nu}}{\partial x^{\beta'}} g^{(1)}_{\mu\nu} \\ &= \delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} g^{(1)}_{\mu\nu} - \partial_{\alpha}\chi^{\mu}\eta_{\mu\beta} - \partial_{\beta}\chi^{\nu}\eta_{\nu\alpha} \\ &= g^{(1)}_{\alpha\beta} - \partial_{\alpha}\chi_{\beta} - \partial_{\beta}\chi_{\alpha} \end{aligned} \tag{15.16}$$

[3]We note the index structure of this equation. While the first equality in (15.16) represents the standard general coordinate transformation (with primed indices on both sides of the equation), the primed indices disappear in the subsequent right-hand-sides because of the gauge transformation of (15.12).

where $\chi_{\alpha} = \chi^{\mu}\eta_{\mu\alpha}$. Expressing both sides in term of $h_{\alpha\beta}$, we have the gauge transformation of the perturbation field

$$h_{\alpha'\beta'} = h_{\alpha\beta} - \partial_{\alpha}\chi_{\beta} - \partial_{\beta}\chi_{\alpha} \tag{15.17}$$

which closely resembles the transformation (12.53) for the electromagnetic 4-vector potential $A_{\alpha}(x)$.

15.1.2 The wave equation in the Lorentz gauge

Just as in electromagnetism, one can streamline some calculations by an appropriate choice of gauge conditions. Here this means that a particular choice of coordinates can simplify the field equation formalism for gravitational waves. We are interested in the coordinate system (Problem 15.1) for which the **Lorentz gauge** (also known as the **harmonic gauge**) condition holds:

$$\partial^{\mu}\bar{h}_{\mu\nu} = 0 \tag{15.18}$$

where $\bar{h}_{\mu\nu}$ is the **trace reversed perturbation:**

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{h}{2}\eta_{\mu\nu} \tag{15.19}$$

with a trace of opposite sign, $\bar{h}^{\mu}_{\mu} \equiv \bar{h} = -h$. From (15.18) and (15.19), we have the **Lorentz gauge** relation $\partial^{\mu}h_{\mu\nu} = \frac{1}{2}\partial_{\nu}h$, which implies, in (15.7) and (15.8), a simplified Ricci tensor $R^{(1)}_{\mu\nu} = -\frac{1}{2}\Box h_{\mu\nu}$, and Ricci scalar $R^{(1)} = -\frac{1}{2}\Box h$. This turns the linearized Einstein equation (15.10) into the form of a standard wave equation:[4]

$$\Box\bar{h}_{\mu\nu} = \frac{16\pi G_{\mathrm{N}}}{c^4} T^{(0)}_{\mu\nu}. \tag{15.20}$$

[4]For a discussion of the Schwarzschild exterior solution for this linearized GR field equation, see Problem 15.2.

Table 15.1 Analog between the electromagnetic and linearized gravitational field theory.

	Electromagnetism	**Linearized gravity**
Source	j^{μ}	$T^{\mu\nu}$
Conservation law	$\partial_{\mu} j^{\mu} = 0$	$\partial_{\mu} T^{\mu\nu} = 0$
Field	A_{μ}	$h_{\mu\nu}$
Gauge transformation	$A_{\mu} \to A_{\mu} - \partial_{\mu}\chi$	$h_{\mu\nu} \to h_{\mu\nu} - \partial_{\mu}\chi_{\nu} - \partial_{\nu}\chi_{\mu}$
Preferred gauge (Lorentz gauge)	$\partial^{\mu} A_{\mu} = 0$	$\partial^{\mu}\bar{h}_{\mu\nu} = 0$ $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}$
Field equation in the preferred gauge	$\Box A_{\mu} = \frac{4\pi}{c} j_{\mu}$	$\Box\bar{h}_{\mu\nu} = \frac{16\pi G_N}{c^4} T_{\mu\nu}$

One can also view this as the equation for the metric field with the energy–momentum tensor being the source of the field. Its retarded solution, expressed as a spatial integral over the source, is

$$\bar{h}_{\mu\nu}(\mathbf{x}, t) = \frac{4G_{\mathrm{N}}}{c^4} \int d^3\mathbf{x}' \frac{T^{(0)}_{\mu\nu}\left(\mathbf{x}', t - \left|\mathbf{x} - \mathbf{x}'\right|/c\right)}{|\mathbf{x} - \mathbf{x}'|}, \tag{15.21}$$

which is certainly compatible with the gauge condition $\partial^{\mu}\bar{h}_{\mu\nu} = 0$ because of the energy momentum conservation (15.11).

To reiterate, in this linear approximation of the Einstein theory, the metric perturbation $h_{\mu\nu}$ may be regarded as the symmetric field of gravity waves propagating in the background of a flat spacetime. A comparison of the linearized Einstein theory with the familiar electromagnetic equations can be instructive. Such an analog is presented in Table 15.1.

15.2 Plane waves and the polarization tensor

We shall first consider the propagation of a gravitational wave in vacuum. Such ripples in the metric can always be regarded as a superposition of plane waves. A gravity wave has two independent polarization states. Their explicit form will be displayed in a particular coordinate system, the transverse-traceless (T T) gauge.

Plane waves

The linearized Einstein equation in vacuum, (15.20) with $T^{(0)}_{\mu\nu} = 0$, is $\Box\bar{h}_{\mu\nu} = 0$. Because the trace $\bar{h} = -h$ satisfies the same wave equation, we also have, from applying the $\Box$ operator to (15.19),

$$\Box h_{\mu\nu} = 0. \tag{15.22}$$

Consider the plane wave solution in the form of

$$h_{\mu\nu}(x) = \epsilon_{\mu\nu} e^{ik_{\alpha}x^{\alpha}} \tag{15.23}$$

where $\epsilon_{\mu\nu}$, the polarization tensor of the gravitational wave, is a set of constants forming a symmetric tensor, $\epsilon_{\mu\nu} = \epsilon_{\nu\mu}$, and k^α is the 4-wavevector $k^\alpha = (\omega/c, \vec{k})$. Substituting (15.23) into (15.22), we obtain $k^2\epsilon_{\mu\nu}e^{ikx} = 0$; thus the wavevector must be a null-vector

$$k^2 = k_\alpha k^\alpha = -\frac{\omega^2}{c^2} + \vec{k}^2 = 0. \tag{15.24}$$

Gravitational waves propagate at the same speed $\omega/|\vec{k}| = c$ as electromagnetic waves. Furthermore, because the wave equation (15.22) is valid only in the coordinates satisfying the Lorentz gauge condition (15.18), the polarization tensor must be "transverse"

$$k^\mu \epsilon_{\mu\nu} = 0. \tag{15.25}$$

The transverse-traceless gauge

There is still some residual gauge freedom left: one can make further coordinate gauge transformations as long as the transverse condition (15.25) is not violated. This requires that the associated gauge vector function χ_μ be constrained by the condition:

$$\Box\chi_\mu = 0. \tag{15.26}$$

Such coordinate freedom can be used to simplify the polarization tensor (see Problem 15.1): one can pick $\epsilon_{\mu\nu}$ to be traceless

$$\epsilon_\mu^{\ \mu} = 0, \tag{15.27}$$

as well as

$$\epsilon_{\mu 0} = \epsilon_{0\mu} = 0. \tag{15.28}$$

This particular choice of coordinates is called the "transverse-traceless gauge," which is a subset of the coordinates satisfying the Lorentz gauge condition.

The 4×4 symmetric polarization matrix $\epsilon_{\mu\nu}$ has 10 independent elements. Equations (15.25), (15.27), and (15.28) which superficially represent nine conditions actually fix only eight parameters because part of the transversality condition (15.25), $k^\mu\epsilon_{\mu 0} = 0$, is trivially satisfied by (15.28). Thus $\epsilon_{\mu\nu}$ has only two independent elements. Namely, the gravitational wave has two independent polarization states. Let us display them. Consider a wave propagating in the z direction $k^\alpha = (\omega, 0, 0, \omega)/c$, the transversality condition (15.25), together with (15.28), implies that $\omega\epsilon_{3\nu} = 0$, or $\epsilon_{3\nu} = \epsilon_{\nu 3} = 0$. Together with the conditions (15.27) and (15.28), the metric perturbation has the form

$$h_{\mu\nu}(z, t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i\omega(z-ct)/c}. \tag{15.29}$$

The two polarization states can be taken to be

$$\epsilon^{\mu\nu}_{(+)} = h_+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \epsilon^{\mu\nu}_{(\times)} = h_\times \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with h_+ and $h_\times$ being the respective "plus" and "cross" amplitudes.

15.3 Detection of gravitational waves

The coordinate-independent feature of any gravitational field is its tidal effect. Thus, the detection of gravitational waves involves the recording of minute changes in the relative positions of a set of test particles. In this section, we shall first deduce the oscillatory pattern of such displacements, then briefly describe the principle underlying the gravitational wave interferometer as a detector of such ripples in spacetime.

15.3.1 Effect of gravitational waves on test particles

Consider a free particle before its encounter with a gravitational wave. It is at rest with a 4-velocity $U^\mu = (c, 0, 0, 0)$. The effect of the gravitational wave on this test particle is determined by the geodesic equation $dU^\mu/d\tau + \Gamma^\mu_{\nu\lambda} U^\nu U^\lambda = 0$. Since only U^0 is non-vanishing at the beginning, it reduces to an expression for the initial acceleration of $(dU^\mu/d\tau)_0 = -c^2\Gamma^\mu_{00}$, which vanishes because the Christoffel symbols are $\Gamma^\mu_{00} = 0$ in the TT gauge.[5] The particle is stationary with respect to the chosen coordinate system—the TT gauge coordinate labels stay attached to the particle. Thus one cannot discover any gravitational field effect on a single particle. This is compatible with our expectation, based on the equivalence principle (EP), that gravity can always be transformed away at a point by an appropriate choice of coordinates. We need to examine the relative motion of at least two particles in order to detect the oncoming change in the curvature of spacetime.

Consider the effect of a gravitational wave with "plus-polarization" $\epsilon^{\mu\nu}_{(+)}$ on two test particles at rest: one at the origin and other located at an infinitesimal distance ξ away on the x axis, hence at an infinitesimally small separation $dx^\mu = (0, \xi, 0, 0)$. Using the expression in (15.29), this translates into a proper separation of

$$\begin{aligned} ds &= \sqrt{g_{\mu\nu}dx^\mu dx^\nu} = \sqrt{g_{11}}\xi \simeq \left[\eta_{11} + \frac{1}{2}h_{11}\right]\xi \\ &= \left[1 + \frac{1}{2}h + \sin\omega\,(t - z/c)\right]\xi \end{aligned} \tag{15.30}$$

showing that the proper distance does change with time. Similarly for two particles separated along the y axis, $dx^\mu = (0, 0, \xi, 0)$, the effect of the

[5]The connection $\Gamma^\mu_{00} = \eta^{\mu\nu}(\partial_0 h_{\nu 0} + \partial_0 h_{0\nu} - \partial_\nu h_{00})/2 = 0$ because the metric perturbation $h_{\mu\nu}$ has, in the TT gauge, polarization components of $\epsilon_{\nu 0} = \epsilon_{0\nu} = \epsilon_{00} = 0$. The vanishing of the initial acceleration means that the particle will be at rest a moment later. Repeating the same argument for later moments, we find the particle at rest for all times. In this way we conclude $dU^\mu/d\tau = 0$.

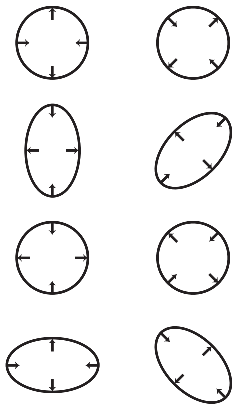

Fig. 15.1 Tidal force effects on a circle of test particles due to gravitational waves in (a) the plus-polarization, and (b) the cross-polarization states.

gravitational wave is to alter the separation according to

$$ds = \left[1 - \frac{1}{2}h_+ \sin\omega\,(t - z/c)\right]\xi. \tag{15.31}$$

Thus, the separation along the x direction is elongated while along the y direction it is compressed. There is no change in the longitudinal separation along the z direction. Just like the electromagnetic waves, gravitational radiation is a transverse field. To better exhibit this pattern of relative displacement we illustrate in Fig. 15.1(a) the effect of a plus-polarized wave, but instead of impinging on two particles as discussed above, acting on a set of test particles when the second particle is replaced by a circle of particles with the first test particle at the center. The outcome that generalizes (15.30) and (15.31) is shown through the wave's one cycle of oscillation.

The effect of a wave with cross-polarization $\epsilon^{\mu\nu}_{(\times)}$ on two particles with differential intervals of $dx^\mu = (0, 1, \pm 1, 0)\xi/\sqrt{2}$ alters the proper separation as $ds = 1 \pm \frac{1}{2}h \times \sin\omega(t - z/c)\xi$. The generalization to a circle of particles through one cycle of oscillation is shown in Fig. 15.1(b), which is just a 45° rotation of the plus-polarized wave result of Fig. 15.1(a). While the two independent polarization directions of an electromagnetic wave are at 90° from each other, those of a gravity wave are at 45°. This is related to the feature that, in the dual description of the wave as streaming particles, the associated particles of these waves have different intrinsic angular momenta: the photon has spin 1 while the graviton has spin 2. It is also instructive to compare the tidal force effects on such test-particles' relative displacement in response to an oncoming oscillatory gravitational field to that of a static gravitational field as discussed in Section 6.3.1.

15.3.2 Gravitational wave interferometers

A gravitational wave can be thought of as a propagating metric, affecting distance measurements. Thus, as a wave passes through, the separation s between two test masses changes with time. Gravitational interaction is very weak. The longitudinal and transverse separation of test particles discussed above is expected to be tiny. Before any detailed calculation (such as the one given in Section 15.4), it is useful to have some idea of the size of the expected gravitational wave signal. Here we give an estimate of the fractional change of separation, called the **strain** $\sigma = (\delta s)/s$, by a "hand-waving" argument.

The separation between two test masses are is by the equation of geodesic deviation (Problems 14.4 and 14.5), but we shall estimate it by using the simpler Newtonian deviation equation of (6.32), which expresses the acceleration per unit separation by the second derivative of the gravitational potential. We assume that the relativistic effect can be included by a multiplicative factor. The Newtonian potential for a spherical source is $\Phi = -G_\mathrm{N}Mr^{-1}$. A gravitational wave propagating in the z direction is a disturbance in the gravitational field:

$$\delta\Phi = -\psi\frac{G_\mathrm{N}M}{r}\sin(kz - \omega t), \tag{15.32}$$

where $k = \omega/c$. A dimensionless factor of ψ has been inserted to represent the relativistic correction. The second derivative can be approximated by

$$\frac{\partial^2}{\partial z^2}\delta\Phi = \psi\frac{G_N M}{rc^2}\omega^2 \sin(kz - \omega t), \tag{15.33}$$

where we have dropped subleading terms coming from differentiation of the r^{-1} factor. This being the acceleration as given in (6.32), the strain amplitude (for the time interval ω^{-1}) is then given by

$$\sigma = \frac{\delta s}{s} = \psi\frac{G_N M}{rc^2}. \tag{15.34}$$

A similar approximation of the radiation formula (15.61) to be derived below suggests the relativistic correction factor ψ as being the nonrelativistic velocity squared $(v/c)^2$ of the source. The first generation of gravitational wave interferometers have been set up with the aim of detecting gravitational wave emission by neutron stars from the richest source of galaxies in our neighboring part of the universe, the Virgo Cluster, at $r \approx 15$ Mpc distance away. Thus, even for a sizable $\psi = O(10^{-1})$ from a solar mass source $M = M_\odot$ the expected strain is only $\sigma = O(10^{-21})$. For two test masses separated by a distance of 10 km the gravity wave induced separation is still one hundredth of a nuclear size dimension. This shows that spacetime is a very stiff medium, as a large amount of energy can still bring about a tiny disturbance in the spacetime metric. This fact poses great challenges to experimental observation of gravitational waves.

The above discussion makes it clear that one needs to design sensitive detectors to measure the minute length changes between test masses over long distances. Several detectors have been constructed based on the Michelson interferometer configuration (Fig. 15.2). The test masses are mirrors suspended to isolate them from external perturbation forces. Light from a laser source is divided into the two arms by a beam splitter. The light entering into an arm of length L is reflected back and forth in a Fabry–Perot cavity for n times so that the optical length is greatly increased and the storage time is $n(L/c) = \Delta t_n$. The return light beams from the two arms are combined after they pass through the beam splitter again. By choosing the path length properly, the optical electric field can be made to vanish (destructive interference) at the photodetector. Once adjusted this way, a stretch in one arm and a compression

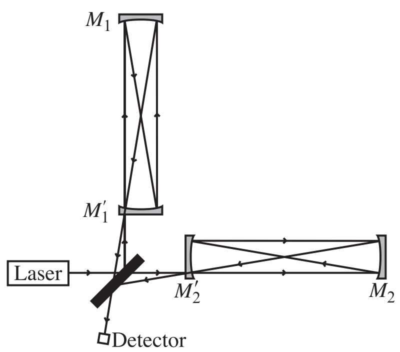

Fig. 15.2 Schematic diagram for gravitational wave Michelson interferometer. The four mirrors $M_{1,2}$, $M'_{1,2}$ and the beam splitter mirror are freely suspended. The two arms are optical cavities that increase the optical paths by many factors. A minute length change of the two arms, one expands and the other contracts, will show up as changes in fringe pattern of the detected light.

in the other, when induced by the polarization of a passing gravitational wave, will change the optical field at the photodetector in proportion to the product of the field times the wave amplitude. Such an interferometer should be uniformly sensitive to wave frequencies less than $\frac{1}{4}\Delta t_n^{-1}$ (and a loss of sensitivity to higher frequencies). The basic principle to achieve high sensitivity is based on the idea that most of the perturbing noise forces are independent of the baseline lengths while the gravitational-wave displacement grows with the baseline.

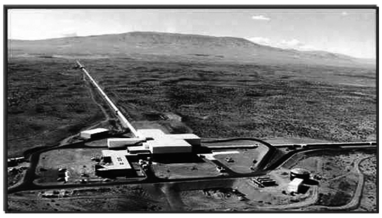

Fig. 15.3 LIGO Hanford Observatory in Washington state.

The Laser Interferometer Gravitational Observatory (LIGO) comprises of two sites: one at the Hanford Reservation in Central Washington (Fig. 15.3) housing two interferometer one 2 km- and another 4 km-long arms, while the other site is at Livingston Parish, Louisiana. The three interferometers are being operated in coincidence so that the signal can be confirmed by data from all three sites. Other gravitational wave interferometers in operation are the French/Italian VIRGO project, the German/Scottish GEO project, and the Japanese TAMA project. Furthermore, study is underway both at the European Space Agency and NASA for the launching of three spacecraft placed in solar orbit with one AU radius, trailing the earth by 20°. The spacecraft are located at the corners of an equilateral triangle with sides 5×10^6 km long. The Laser Interferometer Space Antenna (LISA) consists of single-pass interferometers, set up to observe a gravitational wave at low frequencies (from 10^{-5} to 1 Hz). This spectrum range is expected to include signals from several interesting interactions of black holes at cosmological distances.[6]

[6]Besides planning and building ever large-scale gravitational wave detectors, a major effort by the theoretical community in relativity is involved in the difficult task of calculating wave shapes in various strong gravity situations (e.g. neutron-star/neutron-star collision, black hole mergers, etc.) to guide the detection and comparison of theory with experimental observations.

15.4 Emission of gravitational waves

Although, as of this writing, there has not been any generally accepted proof for a direct detection of gravitational wave, there is nevertheless convincing, albeit indirect, evidence for the existence of such waves as predicted by Einstein's theory. Just as any shaking of electric charges produces electromagnetic waves, a shaking of masses will result in the generation of gravitational wave, which carries away energy. In this section we present the relativistic binary pulsar system showing that the decrease of its orbital period due to gravitational wave radiation is in excellent agreement with what is predicted by general relativity. Since such a calculation is somewhat lengthy, we provide an outline of the steps required for its derivation in Box 15.1.

Box 15.1 The steps for calculating the energy loss by a radiating source

- The rate of energy loss *dE/dt* due to gravitational radiation is the area integral over the radiation flux. This flux (energy per unit area and per unit time) should be the energy density (energy per unit volume) of the radiation gravitational field times the radiation velocity, $f = ct_{00}$, where t_{00} is the (0, 0) component of the energy–momentum tensor $t_{\mu\nu}$ of the radiation field.

- Just as the energy–momentum tensor of an electromagnetic field can be expressed directly in terms of the field itself (as in Box 12.5), $t_{\mu\nu}$ can be expressed in terms of the gravitational (perturbation) field $h_{\mu\nu}$. In particular, the energy density term will be shown in (15.50) as relating to the time derivative of the spatial components of the perturbation in the TT gauge as

$$t_{00} = \frac{c^2}{32\pi G_{\mathrm{N}}} \left\langle \left(\partial_t h_{ij}^{\mathrm{TT}}\right)\left(\partial_t h_{ij}^{\mathrm{TT}}\right)\right\rangle, \tag{15.35}$$

where $\langle ... \rangle$ represents the averaging over many wavelengths.

- The gravitational field is related to the energy–momentum tensor $T_{\mu\nu}$ of the source through the Einstein field equation. In our case we can find the metric perturbation $h_{\mu\nu}$ by solving the wave equation in terms of the source distribution. The leading term for h_{ij} is given in (15.53):

$$h_{ij} = \frac{2G_{\mathrm{N}}}{c^4 r}\left(\partial_t^2 I_{ij}\right) \tag{15.36}$$

where $I_{ij} = \int d^3\mathbf{x}\rho(\mathbf{x})x_i x_j$, with $\rho(\mathbf{x})$ being the mass density of the source. Namely, it is quadrupole radiation.

- After averaging over all directions, one finds in (15.61) that the radiation power is related to the "reduced quadrupole moments $\tilde{I}_{ij}$" of (15.57) as

$$\frac{dE}{dt} = \frac{G_{\mathrm{N}}}{5c^5}\left\langle \left(\partial_t^3 \tilde{I}_{ij}^{\mathrm{TT}}\right)\left(\partial_t^3 \tilde{I}_{ij}^{\mathrm{TT}}\right)\right\rangle. \tag{15.37}$$

Thus, one can start the calculation of the gravitational radiation loss due to orbiting binary stars by computing, from its equation of motion, the quadrupole moments of such a system. This will be carried out in Section 15.4.3.

15.4.1 Energy flux in linearized gravitational waves

In the linearized Einstein theory, gravitational waves are regarded as small curvature ripples propagating in a background of flat spacetime. But gravity waves, just like electromagnetic waves, carry energy and momentum; they will in turn produce additional curvature in the background spacetime. Thus we should have a slightly curved background and (15.1) should be generalized to

$$g_{\mu\nu} = g_{\mu\nu}^{(b)} + h_{\mu\nu} \tag{15.38}$$

where $g_{\mu\nu}^{(b)} = \eta_{\mu\nu} + O(h^2)$ is the background metric. The Ricci tensor can similarly be decomposed as

$$R_{\mu\nu} = R_{\mu\nu}^{(b)} + R_{\mu\nu}^{(1)} + R_{\mu\nu}^{(2)} + \cdots$$

where $R_{\mu\nu}^{(n)} = O(h^n)$ with $n = 1, 2, \ldots$ Thus, the background curvature $R_{\mu\nu}^{(b)}$ should be of the same order as $R_{\mu\nu}^{(2)}$. In free space the Einstein equation being

$R_{\mu\nu} = 0$, terms corresponding to different orders of metric perturbation on the RHS must vanish separately: $R^{(1)}_{\mu\nu} = 0$ as well as

$$R^{(b)}_{\mu\nu} + R^{(2)}_{\mu\nu} = 0. \tag{15.39}$$

The energy–momentum tensor $t_{\mu\nu}$ carried by the gravity wave provides the slight curvature of the background spacetime. It must therefore be related, at this order, to the background Ricci tensor by way of the Einstein equation (14.26):

$$R^{(b)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(b)} = -\frac{8\pi G_{\mathrm{N}}}{c^4}t_{\mu\nu}.$$

That is, $t_{\mu\nu}$ is fixed by $R^{(b)}_{\mu\nu}$, which in turn is related to $R^{(2)}_{\mu\nu}$ by way of (15.39). This allows us to calculate $t_{\mu\nu}$ through the second-order Ricci tensor and scalar

$$t_{\mu\nu} = \frac{c^4}{8\pi G_{\mathrm{N}}}\left(R^{(2)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(2)}\right). \tag{15.40}$$

Before carrying out the calculation of $t_{\mu\nu}$, we should clarify one point: the concept of **local** energy of a gravitational field does not exist. Namely, one cannot specify the gravitational energy at any single point in space. This is so because the energy, being a coordinate-independent function of field, one can always, according to the EP, find a coordinate (the local inertial frame) where the gravity field vanishes locally. Saying it in another way, just as in electromagnetism, we expect the energy density to be proportional to the square of the potential's first derivative. But, according to the flatness theorem, the first derivative of the metric vanishes in the local inertial frame. Thus we cannot speak of gravity's local energy. Nevertheless, one can associate an effective energy–momentum tensor with the gravitational field of finite volume. Specifically, we can average over a spatial volume that is much larger than the wavelength of the relevant gravitational waves to obtain

$$t_{\mu\nu} = \frac{c^4}{8\pi G_N}\left[\left\langle R^{(2)}_{\mu\nu}\right\rangle - \frac{1}{2}\eta_{\mu\nu}\left\langle R^{(2)}\right\rangle\right] \tag{15.41}$$

where $\langle\ldots\rangle$ stands for the average over many wave cycles.

Let us calculate the energy flux carried by a linearly polarized plane wave, say the h_+ state, propagating in the z direction. The metric and its inverse, accurate up to first order in the perturbation, in the TT gauge can be written as

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1+\tilde{h}_+ & 0 & 0 \\ 0 & 0 & 1-\tilde{h}_+ & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1-\tilde{h}_+ & 0 & 0 \\ 0 & 0 & 1+\tilde{h}_+ & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{15.42}$$

where

$$\tilde{h}_+ = h_+ \cos\left[\omega\left(t - z/c\right)\right]. \tag{15.43}$$

To obtain the energy–momentum tensor of the gravity wave by way of $R^{(2)}_{\mu\nu}$ as in (15.40), we need first to calculate the Christoffel symbols by differentiating the metric of (15.42). It can be shown (Problem 15.4-a) that the nonvanishing elements are

$$\Gamma^1_{10} = \Gamma^1_{01} = \Gamma^0_{11} = \frac{1}{2}\left(\partial_0 \tilde{h}_+ - \tilde{h}_+ \partial_0 \tilde{h}_+\right) \tag{15.44}$$

and, similarly,

$$\Gamma^1_{13} = \Gamma^1_{31} = -\Gamma^3_{11} = -\frac{1}{2}(\partial_0 \tilde{h}_+ - \tilde{h}_+ \partial_0 \tilde{h}_+). \tag{15.45}$$

The Riemann tensor has the structure of $(\partial\Gamma + \Gamma\Gamma)$. Since we are only interested in an $O(h^2)$ calculation, the above $\tilde{h}_+ \partial_0 \tilde{h}_+$ factor in the Christoffel symbols can only enter in the $\partial\Gamma$ terms, leading to the time-averaged term of $\langle \tilde{h}_+ \partial_0 \tilde{h}_+ \rangle \propto \langle \sin 2\omega(t - z/c) \rangle = 0$. Hence we will drop the $\tilde{h}_+ \partial_0 \tilde{h}_+$ terms in (15.44) and (15.45), and calculate the (averaged) curvature tensor in (13.58) by dropping the $\langle \partial\Gamma \rangle$ factors,

$$\left\langle R^{(2)}_{\mu\nu} \right\rangle = \left\langle \Gamma^\alpha_{\alpha\lambda} \Gamma^\lambda_{\mu\nu} - \Gamma^\alpha_{\mu\lambda} \Gamma^\lambda_{\alpha\nu} \right\rangle. \tag{15.46}$$

A straightforward calculation (Problem 15.4-b) shows that

$$R^{(2)}_{11} = R^{(2)}_{22} = 0 \quad \text{and} \quad R^{(2)}_{00} = R^{(2)}_{33} = \frac{1}{2}\left(\partial_0 \tilde{h}_+\right)^2 \tag{15.47}$$

leading to a vanishing Ricci scalar

$$R^{(2)} = \eta^{\mu\nu} R^{(2)}_{\mu\nu} = -R^{(2)}_{00} + R^{(2)}_{11} + R^{(2)}_{22} + R^{(2)}_{33} = 0. \tag{15.48}$$

In particular, the effective energy density of the gravitational plane wave in the plus polarization state, as given by (15.41) and (15.47), yields the first term on the RHS of the following relation:

$$t_{00} = \frac{c^4}{16\pi G_{\mathrm{N}}} \left\langle \left(\partial_0 \tilde{h}_+\right)^2 + \left(\partial_0 \tilde{h}_\times\right)^2 \right\rangle \tag{15.49}$$

where we have also added, the second term on the RHS, the corresponding contribution from the cross-polarization state. If we choose to write the transverse traceless metric perturbation as $\tilde{h}_+ \equiv h^{\mathrm{TT}}_{11} = -h^{\mathrm{TT}}_{22}$ and $\tilde{h}_\times \equiv h^{\mathrm{TT}}_{12} = h^{\mathrm{TT}}_{21}$ and $h^{\mathrm{TT}}_{3i} = 0$ (with $i = 1, 2, 3$), we then have

$$\left\langle \left(\partial_0 \tilde{h}_+\right)^2 + \left(\partial_0 \tilde{h}_\times\right)^2 \right\rangle = \left\langle \left(\partial_t h^{\mathrm{TT}}_{ij}\right) \left(\partial_t h^{\mathrm{TT}}_{ij}\right) \right\rangle / \left(2c^2\right).$$

For a wave travelling at the speed c the energy flux being related to the density by $f = ct_{00}$, hence can be expressed it in terms of the metric perturbation as

$$f = \frac{c^3}{32\pi G_{\mathrm{N}}} \left\langle \left(\partial_t h^{\mathrm{TT}}_{ij}\right) \left(\partial_t h^{\mathrm{TT}}_{ij}\right) \right\rangle \tag{15.50}$$

with repeated indices summed over. It is useful to recall the counterpart in the more familiar electromagnetism. The EM flux is given by the field energy density (multiplied by c) which is proportional to the square of the field, or the square of the time derivatives of the (vector) potential. Equation (15.50) shows that a gravitational wave is just the same, with the proportionality constant built out of c and G_{N}. One can easily check that c^3/G_{N} has just the right units (energy times time per unit area). It is a large quantity, again reflecting the stiffness of spacetime—a tiny disturbance in the metric corresponds to a large energy flux.

15.4.2 Energy loss due to gravitational radiation emission

In the previous subsection we have expressed the energy flux of a gravitational wave in terms of the metric perturbation $h_{ij} = g_{ij} - g_{ij}^{(b)}$. Here we will relate h_{ij} to the source of a gravitational wave by way of the linearized Einstein equation (15.21).

Calculate the wave amplitude due to quadrupole moments

We shall be working in the long-wavelength limit for a field-point far away from the source. Let D be the dimension of the source. This limit corresponds to

$$\begin{aligned} r &\gg D \qquad \text{large distance from source} \\ \lambda &\gg D \qquad \text{long wavelength.} \end{aligned}$$

In such a limit we can approximate the integral over the energy–momentum source in (15.21) as

$$\int d^3\mathbf{x}' \frac{T_{\mu\nu}\left(\mathbf{x}', t - \left|\mathbf{x} - \mathbf{x}'\right|/c\right)}{|\mathbf{x} - \mathbf{x}'|} \longrightarrow \frac{1}{r}\int d^3\mathbf{x}' T_{\mu\nu}\left(\mathbf{x}', t - \frac{r}{c}\right) \tag{15.51}$$

because in the long wave limit[7] the harmonic source $T_{\mu\nu} \propto \cos \omega t - 2\pi|\mathbf{x} - \mathbf{x}'|/\lambda$ will not change much when integrated over the source. To calculate the energy flux through (15.50) we have, from (15.21) and (15.51),

$$h_{ij}(\mathbf{x}, t) = \frac{4G_\mathrm{N}}{c^4 r}\int d^3\mathbf{x}' T_{ij}\left(\mathbf{x}', t - \frac{r}{c}\right), \tag{15.52}$$

where we have not distinguished between h_{ij} and $\bar{h}_{ij}$ as they are the same in the TT gauge.

To calculate $\int d^3\mathbf{x}' T_{ij}(\mathbf{x}')$ we find it convenient to convert it into a second mass moment[8] by way of the energy–momentum conservation relation $\partial_\mu T^{\mu\nu} = 0$, and express the integral as the time derivative of the moment integral of the $(0, 0)$ component of the energy–momentum tensor:

$$h_{ij}(\mathbf{x}, t) = \frac{2G_\mathrm{N}}{c^4 r}\frac{\partial^2}{\partial t^2} I_{ij}\left(t - \frac{r}{c}\right) \tag{15.53}$$

where I_{ij} is the second mass moment, after making the Newtonian approximation of the energy density as $T_{00} = \rho c^2$ with $\rho(\mathbf{x})$ being the mass density, given by

$$I_{ij} = \int d^3\mathbf{x}\rho\left(\mathbf{x}\right) x_i x_j. \tag{15.54}$$

We have already explained that, just as in the electromagnetic case, there is no monopole radiation (Birkhoff's theorem). But unlike electromagnetism,

[7] The long-wavelength approximation means small (D/λ) and small ωt as $t \sim D/c$ and $\omega \sim c/\lambda$.

[8] Differentiating the conservation conditions $\partial_0\partial_\mu T^{\mu 0} = 0$ leads to

$$\frac{\partial^2 T^{00}}{c^2\partial t^2} = -\frac{\partial^2 T^{i0}}{c\partial t\partial x^i} = -\frac{\partial}{\partial x^i}\frac{\partial T^{0i}}{c\partial t}.$$

We can apply the conservation relation $\partial_0 T^{0i} + \partial_j T^{ij} = 0$ one more time to get

$$\frac{\partial^2 T^{00}}{c^2\partial t^2} = +\frac{\partial^2 T^{ij}}{\partial x^i\partial x^j}.$$

Multiply both sides by $x^k x^l$ and integrate over the source volume

$$\begin{aligned} \frac{\partial^2}{c^2\partial t^2}\int d^3\mathbf{x} T^{00} x^k x^l &= \int d^3\mathbf{x}\frac{\partial^2 T^{ij}}{\partial x^i\partial x^j} x^k x^l \\ &= 2\int d^3\mathbf{x} T^{kl}. \end{aligned}$$

To reach the last equality we have performed two integrations-by-parts and discarded the surface terms because the source dimension is finite.

there is also no gravitational dipole radiation because the second-order time derivative of the dipole moment

$$\partial_t^2 \mathbf{d} = \int d^3\mathbf{x} \rho\,(\mathbf{x})\,\dot{\mathbf{v}} = 0 \tag{15.55}$$

is the total force on the system. It vanishes for an isolated system (reflecting momentum conservation). Thus the leading gravitational radiation must be quadrupole radiation, as shown in (15.53) and (15.54).

Summing over the flux in all directions in the TT gauge

The energy flux we need to calculate is, according to (15.50), directly related to the metric perturbation h_{ij} in the traceless-transverse gauge, while the result (15.53) shows that h_{ij} is given by the quadrupole moment I_{ij}. To have the mass moment with the same traceless and transverse structure as the metric perturbation, h_{ij}^{TT}, we must apply the traceless-transverse projection operator onto the mass moment of (15.54). Consider a plane wave propagating in an arbitrary direction, specified by the unit vector $\vec{n} = \vec{r}/r$. The projection operator that imposes the transversality condition is

$$\Pi_{ij} = \delta_{ij} - n_i n_j, \tag{15.56}$$

clearly satisfying the condition $n_i \Pi_{ij} = 0$. As it turns out, the algebra will be simplified if we work with the "reduced mass moment" by subtracting out a term proportional to the trace $I = \delta_{ij} I_{ij}$:

$$\tilde{I}_{ij} = I_{ij} - \frac{1}{3}\delta_{ij} I, \tag{15.57}$$

which is traceless, $\delta_{ij}\tilde{I}_{ij} = 0$. However, one finds that the resultant projection $\Pi_{il}\Pi_{jl}\tilde{I}_{kl}$ is still not traceless. It is not too difficult to find (Problem 15.5) the traceless-transverse reduced mass moment to be

$$\tilde{I}_{ij}^{\mathrm{TT}} = \Pi_{ik}\Pi_{jl}\tilde{I}_{kl} - \frac{1}{2}\Pi_{ij}\Pi_{kl}\tilde{I}_{kl}. \tag{15.58}$$

It is then straightforward to find (Problem 15.6) that

$$\tilde{I}_{ij}^{\mathrm{TT}}\tilde{I}_{ij}^{\mathrm{TT}} = \frac{1}{2}\left[2\tilde{I}_{ij}\tilde{I}_{ij} - 4\tilde{I}_{ik}\tilde{I}_{il}n_k n_l + \tilde{I}_{ij}\tilde{I}_{kl}n_i n_j n_k n_l\right]. \tag{15.59}$$

To calculate the total power emitted by the source, we need to integrate over the flux for a wave propagating out in **all** directions. We obtain[9]

$$\int \frac{1}{2}\left[2\tilde{I}_{ij}\tilde{I}_{ij} - 4\tilde{I}_{ik}\tilde{I}_{il}n_k n_l + \tilde{I}_{ij}\tilde{I}_{kl}n_i n_j n_k n_l\right] d\Omega$$
$$= 2\pi\left(2 - \frac{4}{3} + \frac{2}{15}\right)\tilde{I}_{ij}\tilde{I}_{ij} = \frac{8\pi}{5}\tilde{I}_{ij}\tilde{I}_{ij}. \tag{15.60}$$

Integrating the flux (15.50), with the wave amplitude h_{ij} given by (15.53), over all directions by using the result of (15.60) we arrive at the expression for the total luminosity

$$\frac{dE}{dt} = \int f \cdot r^2 d\Omega = \frac{G_{\mathrm{N}}}{5c^5}\left\langle\left(\partial_t^3 \tilde{I}_{ij}^{\mathrm{TT}}\right)\left(\partial_t^3 \tilde{I}_{ij}^{\mathrm{TT}}\right)\right\rangle. \tag{15.61}$$

[9] For this we need to use the formulas

$$\int d\Omega = 4\pi$$
$$\int n_k n_l d\Omega = \frac{4\pi}{3}\delta_{kl}$$
$$\int n_i n_j n_k n_l d\Omega = \frac{4\pi}{15}\left(\delta_{ij}\delta_{kl} + \delta_{kj}\delta_{il} + \delta_{ik}\delta_{jl}\right).$$

These integration results are easy to understand: the only available symmetric tensor that is invariant under rotation is the Kronecker delta δ_{ij}. After fixing the tensor structure of the integrals, the coefficients in front, $4\pi/3$ and $4\pi/15$, can be obtained by contracting the indices on both sides and using the relation $\delta_{ij}\delta_{ij} = 3$.

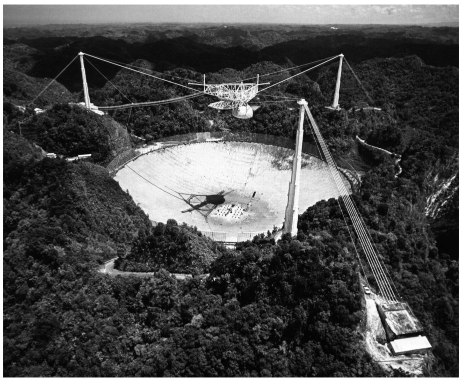

Fig. 15.4 The Arecibo Radio telescope. (Courtesy of the NAIC–Arecibo Observatory, a facility of the NSF.)

Let us recapitulate: the energy carried away by a gravitational wave must be proportional to the square of the time-derivative of the wave amplitude (recall the Poynting vector), which is the second derivative of the quadrupole moment, cf. (15.53). The energy flux falls off like r^{-2}. To get the total luminosity by integrating over a sphere of radius r, the dependence of radial distance disappears. The factor of $G_{\rm N}c^{-5}$ must be present on dimensional grounds. The detailed calculation fixes the proportional constant of $1/5$ and we have the gravitational wave luminosity in the quadrupole approximation displayed above.

15.4.3 Hulse–Taylor binary pulsar

A radio survey, using the Arecibo Radio Telescope in Puerto Rico (Fig. 15.4), for pulsars in our galaxy made by Russel Hulse and Joseph Taylor discovered the unusual system PSR 1913 + 16. Observations made since 1974 allowed them to check GR to great precision including the verification of the existence of gravitational waves as predicted in Einstein's theory.

From the small changes in the arrival times of the pulses recorded in the past decades a wealth of properties of this binary system can be extracted. This is achieved by modeling the orbit dynamics and expressing these in terms of the arrival time of the pulse. Different physical phenomena (such as bending of the light, periastron advance, etc.) are related to the pulse time through different combinations of system parameters. In this way the masses and separation of the stars and the inclination and eccentricity of their orbit can all be deduced (see Table 15.2). It is interesting to note that these two neutron star have just the masses $1.4M_\odot$ of the Chandrasekhar limit (first mentioned in Section 8.3.1).

In this section we shall demonstrate that from these numbers, without any adjustable parameters, we can compute the decrease (decay) of the orbital period due to gravitational radiation by the orbiting binary system. Instead of a full-scale GR calculation, we shall consider the simplified case of two equal mass stars in a circular orbit (Fig. 15.5), as all essential features of gravitational radiation and orbit decay can be easily computed. At the end we then quote the exact GR expression for the pulsar and its companion $M_p \neq M_c$ in an orbit

Table 15.2 Parameters of the Hulse–Taylor binary pulsar system as compiled by Weisberg and Taylor (2003).

Pulsar mass	$M_p = 1.4408 \pm 0.0003\, M_\odot$
Companion mass	$M_c = 1.3873 \pm 0.0003\, M_\odot$
Eccentricity	$e = 0.6171338 \pm 0.000004$
Binary orbit period	$P_\mathrm{b} = 0.322997462727\,\mathrm{d}$
Orbit decay rate	$\dot{P}_\mathrm{b} = (-2.4211 \pm 0.0014) \times 10^{-12}\,\mathrm{s/s}$

with high eccentricity as a straightforward modification of the result obtained by our simplified calculation.

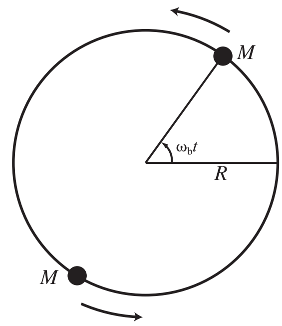

Fig. 15.5 A binary of two equal masses circulating each other in a circular orbit with angular frequency ω_b.

Energy loss due to gravitational radiation

Let us first concentrate on the instantaneous position of one of the binary stars as shown in Fig. 15.5:

$$x_1(t) = R\cos\omega_\mathrm{b}t, \quad x_2(t) = R\sin\omega_\mathrm{b}t, \quad x_3(t) = 0.$$

From this we can calculate the second mass moment according to (15.54),

$$\begin{aligned} I_{11} &= 2MR^2\cos^2\omega_\mathrm{b}t \\ I_{22} &= 2MR^2\sin^2\omega_\mathrm{b}t \\ I_{12} &= 2MR^2\sin\omega_\mathrm{b}t\cos\omega_\mathrm{b}t \end{aligned}$$

leading to the traceless reduced moment as defined in (15.57),

$$\tilde{I}_{ab} = I_{ab} - \frac{1}{2}\delta_{ab}I = I_{ab} - MR^2\delta_{ab}$$

so that

$$\begin{aligned} \tilde{I}_{11} &= MR^2\cos 2\omega_\mathrm{b}t \\ \tilde{I}_{22} &= -MR^2\cos 2\omega_\mathrm{b}t \\ \tilde{I}_{12} &= MR^2\sin 2\omega_\mathrm{b}t. \end{aligned}$$

The quadrupole formula (15.61) for luminosity involves time derivatives. For the simple sinusoidal dependence given above, each derivative just brings down a factor of $2\omega_\mathrm{b}$; together with the averages $\langle\sin^2\rangle = \langle\cos^2\rangle = 1/2$, we obtain the rate of energy loss due to gravitational radiation:

$$\frac{dE}{dt} = \frac{G_\mathrm{N}}{5c^5}(2\omega_\mathrm{b})^6\left\langle \tilde{I}_{11}{}^2 + \tilde{I}_{22}{}^2 + 2\,\tilde{I}_{12}{}^2\right\rangle = \frac{128G_\mathrm{N}}{5c^5}\omega_\mathrm{b}^6 M^2R^4. \qquad (15.62)$$

From energy loss to orbital decay

Energy loss leads to orbital decay, namely the decrease in orbital period P_b of the binary system. We start the calculation of this orbital period change through the relation $(dP_\mathrm{b})/P_\mathrm{b} \propto -(dE)/E$. Again we shall only work out the simpler situation of a binary pair of equal mass M separated by $2R$ in circular motion. The total energy being

$$E = MV^2 - \frac{G_\mathrm{N}M^2}{2R} \qquad (15.63)$$

with velocity determined by the Newtonian equation of motion $MV^2/R = G_N M^2/(2R)^2$ satisfies

$$V^2 = \frac{G_N M}{4R} \tag{15.64}$$

so that the total energy of the binary system (15.63) comes out to be

$$E = -\frac{G_N M^2}{4R}. \tag{15.65}$$

We wish to have an expression of the energy in terms of the orbital period by replacing R using (15.64)

$$R = \frac{G_N M}{4V^2} = \frac{G_N M}{4}\left(\frac{2\pi R}{P_b}\right)^{-2} \quad \text{or} \quad R^3 = \frac{G_N M}{16\pi^2} P_b^2. \tag{15.66}$$

Plugging this back into (15.65), we have

$$E = -M\left(\frac{\pi M G_N}{2}\right)^{2/3} P_b^{-2/3}. \tag{15.67}$$

Through the relation, $dE/E = -\frac{2}{3} dP_b/P_b$, the rate of period decrease $\dot{P}_b \equiv dP_b/dt$ can be related to the energy loss rate

$$\dot{P}_b = -\frac{3P_b}{2E}\left(\frac{dE}{dt}\right). \tag{15.68}$$

Substituting in the expression (15.67) for E in the denominator, (15.62) for (dE/dt) where the wave frequency is given by the orbit frequency $\omega_b = 2\pi/P_b$ and where R is given by (15.66), we obtain the expression for orbital decay rate in this simplified case of two equal masses in circular orbit:

$$\dot{P}_b = -\frac{48\pi}{5c^5}\left(\frac{4\pi G_N M}{P_b}\right)^{5/3}. \tag{15.69}$$

That the orbit for the Hulse–Taylor binary, rather than circular, is elliptical with high eccentricity can be taken into account (Peters and Mathews 1963) with the result involving a multiplicative factor of

$$\frac{1 + (73/24)e^2 + (37/96)e^4}{(1-e^2)^{7/2}} = 11.85681, \tag{15.70}$$

where we have use the observed binary orbit eccentricity as given in Table 15.2. That the pulsar and its companion have slightly different masses, $M_p \neq M_c$, means we need to make the replacement $(2M)^{5/3} \longrightarrow 4M_p M_c (M_p + M_c)^{-1/3}$. The exact GR prediction is found to be

$$\begin{aligned} \dot{P}_{b\,\mathrm{GR}} &= \frac{-192\pi}{5c^5} \frac{1 + (73/24)e^2 + (37/96)e^4}{(1-e^2)^{7/2}} \left(\frac{2\pi G_N}{P_b}\right)^{5/3} \frac{M_p M_c}{(M_p + M_c)^{1/3}} \\ &= -(2.40247 \pm 0.00002) \times 10^{-12}\ \mathrm{s/s}. \end{aligned} \tag{15.71}$$

This is to be compared to the observed value corrected for the galactic acceleration of the binary system and the sun, which also cause a change of orbit

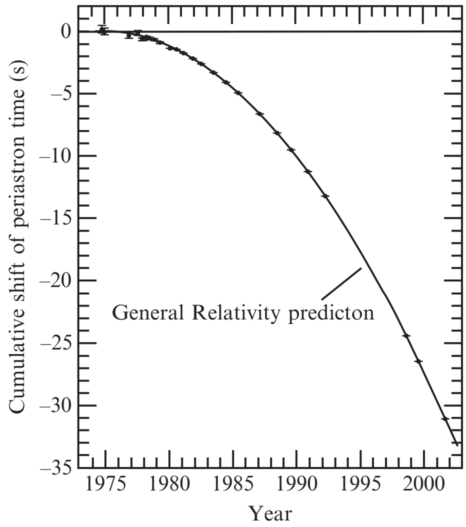

Fig. 15.6 Gravitational radiation damping causes orbital decay of the Hulse–Taylor binary pulsar. Plotted here is the accumulating shift in the epoch of periastron (Weisberg and Taylor, 2003). The parabola is the GR prediction, and observations are depicted by data points. In most cases the measurement uncertainties are smaller than the line widths. The data gap in the 1990s reflects the downtime when the Arecibo Observatory was being upgraded.

period $\dot{P}_{\mathrm{b\,gal}} = -(0.0125 \pm 0.0050) \times 10^{-12}$ s/s. From the measured values given in Table 15.2, we then have

$$\begin{aligned}\dot{P}_{\mathrm{b\,corrected}} &= \dot{P}_{\mathrm{b\,observed}} - \dot{P}_{\mathrm{b\,gal}} \\ &= -(2.4086 \pm 0.0052) \times 10^{-12}\ \mathrm{s/s},\end{aligned} \tag{15.72}$$

in excellent agreement with the theoretical prediction shown in (15.71). This result (Fig. 15.6) provides strong confirmation of the existence of gravitational radiation as predicted by Einstein's theory of general relativity.

With the confirmation of the existence of gravitational radiation according to Einstein's general theory of relativity. The next stage will be the detection of gravitational waves through interferometer observations to confirm the expected wave kinematics, and tests of various strong field situations. But just like all pioneering efforts of fresh ways to observe the universe, gravitational wave observatories will surely discover new phenomena that will deepen and challenge our understanding of astronomy, gravitation and cosmology.

Review questions

1. Give a qualitative discussion showing why one would expect gravitational waves from Einstein's GR theory of gravitation but not from Newton's theory.
2. Why is it important to have a gravitational wave observatory?
3. What approximation is made to have the linearized theory of general relativity? In this framework, how should we view the propagation of gravitational waves?
4. What are the differences and similarities between electromagnetic and gravitational waves?

5. What is a gauge transformation in the linearized theory? What is the Lorentz gauge? Can we make further gauge transformations within the Lorentz gauge?

6. Consider a set of test particles, all of them lying in a circle except one at the center. When a gravitational wave with the + polarization passes through them what will be the relative displacements of these particle going through one period of the wave? How would the relative displacement be different if the polarization is of × type?

7. Give a qualitative argument showing that the wave strain is of the order $\psi G_N M/rc^2$ where $\psi\epsilon$ is a relativistic correction factor typically less than unity. Such a strain would be $O(10^{-21})$ when the wave is generated by a solar mass source in the Virgo Cluster ($r \approx 15$ Mpc) from us.

8. Using what you know of the Poynting vector as the energy flux of an EM wave, guess the form of energy flux in terms of the gravitational wave amplitude. What should be the proportionality constant (up to some numerical constant that can only derived by detailed calculation)?

9. The leading term in gravitational radiation is quadrupole. Why is there no monopole and dipole radiation?

10. Evidence for gravitational waves is obtained by the study of the Hulse–Taylor binary pulsar system. What is being observed? Which results show strong evidence for the existence of gravitation waves as predicted by GR?

Problems

15.1 **Gauge transformations**

(a) Show that the gauge transformation for the trace-reversed perturbation $\bar{h}_{\mu\nu}$

$$\bar{h}'_{\alpha\beta} = \bar{h}_{\alpha\beta} - \partial_\alpha \chi_\beta - \partial_\beta \chi_\alpha + \eta_{\alpha\beta}(\partial\chi)$$

follows from (15.17) and (15.19).

(b) Demonstrate the existence of the Lorentz gauge by showing that, starting with an arbitrary coordinate system where $\partial^\mu \bar{h}_{\mu\nu} \neq 0$, one can always find a new system such that $\partial^\mu \bar{h}'_{\mu\nu} = 0$ with a gauge vector function χ_μ being the solution to the inhomogeneous wave equation $\Box\chi_\nu = \partial^\mu \bar{h}_{\mu\nu}$. This also means that one can make further coordinate transformations within the Lorentz gauge, as long as the associated gauge vector function satisfies the wave equation

$$\Box\chi_\nu = 0. \tag{15.73}$$

(c) The solution to Eq (15.73) may be written as $\chi_\nu = X_\nu e^{ikx}$ where k^α is a null-vector. Show that the four constants X_ν can be chosen so that the polarization tensor in the metric perturbation $h_{\mu\nu}(x)$ is traceless $\epsilon^\mu_{\ \mu} = 0$ and every zeroth component vanishes $\epsilon_{\mu 0} = 0$.

(d) From the results discussed above, show that there are two independent elements in the polarization tensor $\epsilon_{\mu\nu}$.

15.2 **The Schwarzschild solution** Obtain the exterior solution to the linearized Einstein equation (15.20) for a spherical source. Since it is of the form of a standard wave equation, the spherically symmetric solution should have the same form as the $1/r$ potential:

$$\bar{h}_{\mu\nu} = \frac{C_{\mu\nu}}{r}, \tag{15.74}$$

where $C_{\mu\nu}$ is a constant tensor. Show that the Lorentz gauge condition (15.18) implies that every element of this tensor vanishes except C_{00}, which is fixed by its Newtonian limit to be $C_{00} = 2r^*$, twice the Schwarzschild radius. From this you are asked to show that the resultant metric may be written in a spherical coordinate system as

$$g_{\mu\nu} = \operatorname{diag}\left[\left(-1 + \frac{r^*}{r}\right), \left(1 + \frac{r^*}{r}\right), r^2, r^2 \sin^2\theta\right], \tag{15.75}$$

which approximates the Schwarzschild metric of (14.54) up to a correction $O(r^{*2})$. **Beware** of the fact that the linearized theory developed in this chapter, hence Eq. (15.20), assumes a flat space metric in the Cartesian coordinate $\eta_{\mu\nu} = \operatorname{diag}(-1, 1, 1, 1)$. This is the coordinate system you should use to start your discussion; only at later stage should you change it to a spherical one—in order to compare your result with (14.54).

15.3 **Wave effect via the deviation equation** As we have shown in Section 15.3.1, a gravitational wave can only be detected through the tidal effect. Since the equation of geodesic deviation (Problem 14.4) is an efficient description of the tidal force, show that the results of (15.30) and (15.31) can be obtained by using this equation.

15.4 **$\Gamma^{\mu}_{\nu\lambda}$ and $R^{(2)}_{\mu\nu}$ in the TT gauge** Show that the Christoffel symbols of (15.44) and (15.45), as well as the second-order Ricci tensor (15.47), are obtained in the TT gauge with the metric given in (15.42).

15.5 **Trace calculation of $\tilde{I}^{\rm TT}_{ij}$** We claimed that $\tilde{I}^{\rm TT}_{ij}$, as defined by (15.58), is traceless. Prove this by explicit calculation.

15.6 **Derive the relation (15.59)** Carry out the contraction of the two $\tilde{I}^{\rm TT}_{ij}$, defined by (15.58), to show

$$\tilde{I}^{\rm TT}_{ij}\tilde{I}^{\rm TT}_{ij} = \frac{1}{2}2\tilde{I}_{ij}\tilde{I}_{ij} - 4\tilde{I}_{ik}\tilde{I}_{il}n_k n_l + \tilde{I}_{ij}\tilde{I}_{kl}n_i n_j n_k n_l. \tag{15.76}$$