6 GR as a geometric theory of gravity – I

- We first present a **geometric** description of the equivalence-principle physics of gravitational time dilation. In this geometric theory, the metric $g_{\mu\nu}(x)$ plays the role of the relativistic gravitational potential.
- Einstein proposed curved spacetime as the gravitational field. The geodesic equation in spacetime is the GR equation of motion, which is checked to have the correct Newtonian limit.
- At every spacetime point, one can construct a free-fall frame in which gravity is transformed away. However, in a finite-sized region, one can detect the residual tidal force which are second derivatives of the gravitational potential. It is the curvature of spacetime.
- The GR field equation directly relates the mass/energy distribution to spacetime's curvature. Its solution is the metric function $g_{\mu\nu}(x)$, determining the geometry of spacetime.

In Chapter 4 we have deduced several pieces of physics from the empirical principle of equivalence of gravity and inertia. In Chapter 5, elements of the mathematical description of a curved space have been presented. In this chapter, we show how some of equivalence-principle physics can be interpreted as the geometric effects of curved spacetime. Such a study motivated Einstein to propose his general theory of relativity, which is a geometric theory of gravitation, with the equation of motion being the geodesic equation, and the field equation in the form of the curvature being directly given by the mass/energy source fields.

6.1 Geometry as gravity

By a geometric theory, or a geometric description, of any physical phenomenon we mean that the physical measurement results can be attributed directly to the underlying geometry of space and time. This is illustrated by the example we discussed in Section 5.2 in connection with a spherical surface as shown in Fig. 5.2. The length measurements on the surface of a globe are different in different directions: the east-west distances between any pairs of points separated by the same azimuthal angle $\Delta\phi$ become smaller as the pair move away from the equator, while the lengths in the north-south directions for a

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fixed ϕ remain the same. We could, in principle, interpret such results in two equivalent ways:

1. Without considering that the 2D space is curved, we can say that physics (i.e. dynamics) is such that the measuring ruler changed scale when pointing in different directions—much in the same manner as the FitzGerald–Lorentz length contraction of SR was originally interpreted.
2. The alternative description (the "geometric theory") is that we use a standard ruler with a fixed scale (defining the coordinate distance) and the varying length measurements are attributed to the underlying geometry of a curved spherical surface. This is expressed mathematically in the form of a position-dependent metric tensor $g_{ab}(x) \neq \delta_{ab}$.

Einstein's general theory of relativity is a geometric theory of gravity—gravitational phenomena are attributed as reflecting the underlying curved spacetime. An interval ds, invariant with respect to coordinate transformations, is related to the coordinates dx^μ of the spacetime manifold through the metric $g_{\mu\nu}$:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{6.1}$$

The Greek indices range over $(0, 1, 2, 3)$ with $x^0 = ct$ and the metric $g_{\mu\nu}$ is a 4×4 matrix. Observers measure with rulers and clocks; the spacetime manifold not only expresses the spatial relations among events but also their causal structure. For special relativity (SR) we have the geometry of a flat spacetime with a position-independent metric $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. The study of equivalence-principle physics led Einstein to propose that gravity represent the structure of a curved spacetime. GR as a geometric theory of gravity posits that matter and energy cause spacetime to warp $g_{\mu\nu} \neq \eta_{\mu\nu}$, and gravitational phenomena are just the effects of a curved spacetime on a test object.

How did the study of the physics as implied by the equivalence principle (EP) motivate Einstein to propose that the relativistic gravitational field was the curved spacetime? We have already discussed the EP physics of gravitational time dilation—clocks run at different rates at positions having different gravitational potential values $\Phi(\vec{x})$, as summarized in (4.32). This variation of time rate follows a definite pattern. Instead of working with a complicated scheme of clocks running at different rates, this physical phenomenon can be given a geometric interpretation as showing a nontrivial metric, $g_{\mu\nu} \neq \eta_{\mu\nu}$. Namely, a simpler way of describing the same physical situation is by using a stationary clock at $\Phi = 0$ as the standard clock. Its fixed rate is taken to be the time coordinate t. One can then compare the time intervals $d\tau(\vec{x})$ measured by clocks located at other locations (the proper time interval at $\vec{x}$) to this coordinate interval dt. According to EP as stated in (4.38), we should find

$$d\tau(\vec{x}) = \left(1 + \frac{\Phi(\vec{x})}{c^2}\right) dt. \tag{6.2}$$

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The geometric approach says that the measurement results can be interpreted as showing a spacetime with a warped geometry having a metric element of

$$g_{00} = -\left(1 + \frac{\Phi(x)}{c^2}\right)^2 \simeq -\left(1 + \frac{2\Phi(x)}{c^2}\right). \tag{6.3}$$

This comes about because (6.1) reduces down to $ds^2 = g_{00}dx^0dx^0$ for $d\vec{x} = 0$, as appropriate for a proper time interval (the time interval measured in the rest frame, hence no displacement) and the knowledge that the line element is just the proper time interval $ds^2 = -c^2d\tau^2$, leading to the expression

$$(d\tau)^2 = -g_{00}\,(dt)^2\,, \tag{6.4}$$

and the result in (6.3). It states that the metric element g_{00} in the presence of gravity deviates from the flat spacetime value of $\eta_{00} = -1$ because of the presence of gravity. Thus the geometric interpretation of the EP physics of gravitational time dilation is to say that gravity changes the spacetime metric element g_{00} from -1 to an x-dependent function. Gravity warps spacetime—in this case it warps it in the time direction. Also, since g_{00} is directly related to the Newtonian gravitational potential $\Phi(x)$ as in (6.3), we can say that the ten independent components of the spacetime metric $g_{\mu\nu}(x)$ **are** the "relativistic gravitational potentials."

6.1.1 EP physics and a warped spacetime

Adopting a geometric interpretation of EP physics, we find that the resultant geometry has all the characteristic features of a **warped** manifold of space and time: a position-dependent metric, deviations from Euclidean geometric relations, and at every location we can always transform gravity away to obtain a flat spacetime, just as one can always find a locally flat region in a curved space.

Position-dependent metrics As we have discussed in Section 5.2, the metric tensor in a curved space is necessarily position dependent. Clearly, (6.3) has this property. In Einstein's geometric theory of gravitation, the metric function is all that we need to describe the gravitational field completely. The metric $g_{\mu\nu}(x)$ plays the role of relativistic gravitational potentials, just as $\Phi(x)$ is the Newtonian gravitational potential.

Non-Euclidean relations In a curved space Euclidean relations no longer hold (see Section 5.3.2): for example, the sum of the interior angles of a triangle on a spherical surface deviates from 180°, the ratio of the circular circumference to the radius is different from the value of 2π. As it turns out, EP does imply a non-Euclidean relation among geometric measurements. We illustrate this with a simple example. Consider a cylindrical room in high speed rotation around its axis. This acceleration case, according to EP, is equivalent to a centrifugal gravitational field. (This is one way to produce "artificial gravity.") For such a rotating frame, one finds that, because of SR (longitudinal) length contraction, the radius, which is not changed because velocity is perpendicular to the radial direction, will no longer equal the circular circumference of the cylinder divided by 2π (see Fig. 6.1 and Problem 6.3). Thus Euclidean geometry is no longer valid in the presence of gravity. We reiterate this connection: the rotating frame, according to EP, is a frame with gravity; the rotating frame, according to SR length contraction, has a relation between its radius and circumference that is not Euclidean. Hence, we say the presence of gravity brings about non-Euclidean geometry.[1]

[1]Distance measurement in a curved spacetime is discussed in Problem 6.2.

Local flat metric and local inertial frame In a curved space a small local region can always be described approximately as a flat space. A more precise statement is given by the flatness theorem of Section 5.2.2. Now, if we identify our spacetime as the gravitational field, there should be the physics result corresponding to this flatness theorem. This is Einstein's key insight that EP will always allow us to transform gravity away in a local region. In this region, because of the absence of gravity, SR is valid and the metric is the flat Minkowski metric. General relativity has the same local lightcone structure as SR: $ds^2 < 0$ being time-like, $ds^2 > 0$ space-like, and $ds^2 = 0$ light-like. The relation between local flat and local inertial frames will be further explored in Section 6.3, where we show that the spacetime curvature is the familiar tidal force. Given that SR is a theory of flat Minkowski spacetime, and motivated to seek a new theory of gravity with a built-in EP, Einstein put forward the elegant solution of having gravity identified as the structure of curved spacetime.

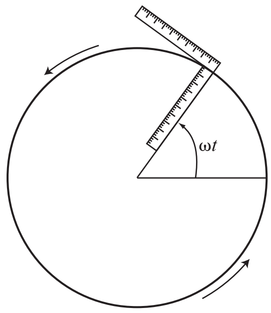

Fig. 6.1 Rotating cylinder with length contraction in the tangential direction but not in the radial direction, resulting in a non-Euclidean relation between circumference and radius.

6.1.2 Curved spacetime as a gravitational field

Recall that a field theoretical description of the interaction between a source and a test particle is a two-step description:

$$\boxed{\text{Source particle}} \xrightarrow[\text{equation}]{\text{Field}} \boxed{\text{Field}} \xrightarrow[\text{motion}]{\text{Equation of}} \boxed{\text{Test particle}}$$

Instead of the source particle acting directly on the test particle through some instantaneous action-at-a-distance force, the source creates a field everywhere, and the field then acts on the test particle locally. The first step is the field equation which, given the source distribution, determines the field everywhere. In the case of electromagnetism it is Maxwell's equation. The second step is provided by the equation of motion, which allows us to find the motion of the test particle, once the field function is known. The electromagnetic equation of motion follows directly from the Lorentz force law.

Newtonian gravitational field

The field equation in Newton's theory of gravity, when written in terms of the gravitational potential $\Phi(x)$, is given by (4.6)

$$\nabla^2\Phi = 4\pi G_N \rho, \tag{6.5}$$

where G_N is Newton's constant, and ρ is the mass density function. The Newtonian theory is not a dynamic field theory as it does not provide a description of time evolution. It is the static limit of some field theory, thus has no field propagation. The Newtonian equation of motion is Eq. (4.8):

$$\frac{d^2\vec{r}}{dt^2} = -\vec{\nabla}\Phi. \tag{6.6}$$

The task Einstein undertook was to find the relativistic generalizations of these two sets of equations (6.5) and (6.6). Since in relativity, space and time are treated on an equal footing, a successful relativistic program will automatically yield a dynamical theory as well.

Relativistic gravitational field

The above discussion suggests that the EP physics can be described in geometric language. The resultant mathematics coincides with that describing a warped spacetime. Thus it is simpler, and more correct, to say that the relativistic gravitational field **is** the curved spacetime. The effect of the gravitational interaction between two particles can be described as the source mass giving rise to a curved spacetime which in turn influences the motion of the test mass. Or, put more strongly, EP requires a metric structure of spacetime and particles follow geodesics in such a curved spacetime.

The possibility of using a curved space to represent a gravitational field can be illustrated with the following example involving a 2D curved surface. Two masses on a spherical surface start out at the equator and move along two geodesic lines as represented by the longitudinal great circles. As they move along, the distance between them decreases (Fig. 6.2). We can attribute this to some attractive force between them, or simply to the curved space causing their trajectory to converge. That is to say, this phenomenon of two convergent particle trajectories can be thought of either as resulting from an attractive tidal force, or from the curvature of the space.[2] Eventually we shall write down the relativistic gravitational equations. In Einstein's approach these differential equations can be thought of as reflecting an underlying warped spacetime.

[2]Further reference to gravitational tidal forces vs. the curvature description of the relative separation between two particle trajectories can be found in Section 6.3.1 when we discuss the Newtonian deviation equation for tidal forces. It has its generalization as the"GR equation of geodesic deviation," given in Chapter 14, see Problems 14.4 and 14.5.

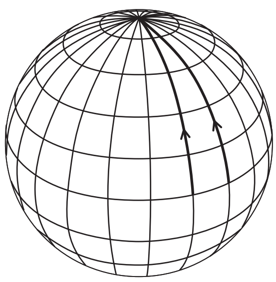

Fig. 6.2 Two particle trajectories with decreasing separation can be interpreted either as resulting from an attractive force or as reflecting the underlying geometry of a spherical surface.

Based on the study of EP phenomenology, Einstein made the conceptual leap (a logical deduction, but a startling leap nevertheless) to the idea that curved spacetime **is** the gravitational field:

$$\boxed{\text{Source}} \xrightarrow[\text{Einstein Field equation}]{} \boxed{\text{Curved specetime}} \xrightarrow[\text{Geodesic equation}]{} \boxed{\text{Test particle}}$$

The mass/energy source gives rise to a warped spacetime, which in turn dictates the motion of the test particle. Plausibly the test particle moves along the shortest and straightest possible curve in the curved manifold. Such a line is the geodesic curve. Hence the GR equation of motion is the geodesic equation (Section 6.2). The GR field equation is the Einstein equation, which relates the mass/energy distribution to the curvature of spacetime (Section 6.3).

While spacetime in SR, like all pre-relativity physics, is fixed, it is dynamic in GR as determined by the matter/energy distribution. GR fulfills Einstein's conviction that "space is not a thing:" the ever changing relation of matter and energy is reflected by an ever changing geometry. Spacetime does not have an independent existence; it is nothing but an expression of the relations among physical processes in the world.

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6.2 Geodesic equation as GR equation of motion

The metric function $g_{\mu\nu}(x)$ in (6.1) describes the geometry of curved spacetime. In GR, the mass/energy source determines the metric function through the field equation. The metric $g_{\mu\nu}(x)$ is the solution of the GR field equation. In this approach, gravity is the structure of spacetime and is not regarded as a force (bringing about acceleration). Thus a test body will move freely in such a curved spacetime, with the equation of motion identified with the geodesic equation.

6.2.1 The geodesic equation recalled

In a geometric theory, the motion of a test body is determined completely by geometry; the GR equation of motion should coincide with the geodesic equation discussed in Chapter 5 on elements of Riemannian geometry. In Section 5.2.1, we have derived the geodesic equation from the property of the geodesic line as the curve with extremum length. We also recall that a point in spacetime is an event and that the trajectory is a worldline (see Box 6.1). The geodesic equation determines the worldline that a test particle will follow under the influence of gravity. The geodesic equation in spacetime is Eq. (5.30) with its Latin indices $a = 1, 2$ (appropriate for the curved 2D space being discussed in Chapter 5) changed into Greek indices $\mu = 0, 1, 2, 3$ with $x^0 = ct$ for a 4D spacetime

$$\frac{d}{d\lambda}\left(g_{\mu\nu}\dot{x}^{\nu}\right) - \frac{1}{2}\frac{\partial g_{\sigma\rho}}{\partial x^{\mu}}\dot{x}^{\sigma}\dot{x}^{\rho} = 0, \tag{6.7}$$

where $x^{\mu} = x^{\mu}(\lambda)$ with λ being the curve parameter, and $\dot{x}^{\mu} \equiv dx^{\mu}/d\lambda$.

We can cast (6.7) into a more symmetric form which will also facilitate our later interpretation (in Section 13.2.2) of the geodesic as the straightest possible curve. Carrying out the differentiation of the first term and noting that the metric's dependence on λ is entirely through $x^{\mu}(\lambda)$:

$$g_{\mu\nu}\frac{d^2x^{\nu}}{d\lambda^2} + \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\frac{dx^{\sigma}}{d\lambda}\frac{dx^{\nu}}{d\lambda} - \frac{1}{2}\frac{\partial g_{\sigma\rho}}{\partial x^{\mu}}\frac{dx^{\sigma}}{d\lambda}\frac{dx^{\rho}}{d\lambda} = 0 \tag{6.8}$$

Box 6.1 The geodesic is the worldline of a test particle

It may appear somewhat surprising to hear that a test particle will follow a "straight line" in the presence of a gravitational field. After all, our experience is just the opposite: when we throw an object, it follows a parabolic trajectory. Was Einstein saying that the parabolic trajectory is actually straight? All such paradoxes result from confusing the 4D spacetime with the ordinary 3D space. The GR equation of motion tells us that a test particle will follow a geodesic line in spacetime—which is not a geodesic line

(*cont.*)

Box 6.1 (*Continued*)

in three-dimensional space. Namely, the worldline of a particle should be a geodesic, which generally does not imply a straight trajectory in the spatial subspace.[3] A simple illustration using the spacetime diagram should make this clear.

Let us consider the case of throwing an object to a height of 10 meters over a distance of 10 meters. Its spatial trajectory is displayed in Fig. 6.3(a). When we represent the corresponding worldline in the spacetime diagram we must plot the time-axis ct also, see Fig. 6.3(b). For the case under consideration, this object takes 1.4 seconds to reach the highest point and another 1.4 seconds to come down. But a 2.8 second time interval will be represented by almost one million kilometers of ct in the spacetime diagram (more than the round trip distance to the moon). When the time axis is stretched out in this way, one then realizes this worldline is very straight indeed, see Fig. 6.3(c). The straightness of this worldline reflects the fact that terrestrial gravity is a very weak field (recall $\Phi_\oplus/c^2 \simeq 10^{-10}$)—it curves the spacetime only a tiny amount. In this case the spacetime is practically flat, and thus the geodesic worldline is very close to a straight line.

[3] In fact this discussion can be used as another motivation for a curved spacetime description of gravity: We know that in the absence of gravity an inertial trajectory corresponds to a straight line in a flat spacetime. In the presence of gravity, a free-falling object traces out a curved trajectory in a flat spacetime diagram. Yet, according to EP this can be viewed as an inertial trajectory with a "straight worldline." This is possible only if the spacetime is curved and locally the space can be approximated by a flat space, as EP implies a **local equivalence** of acceleration and gravity.

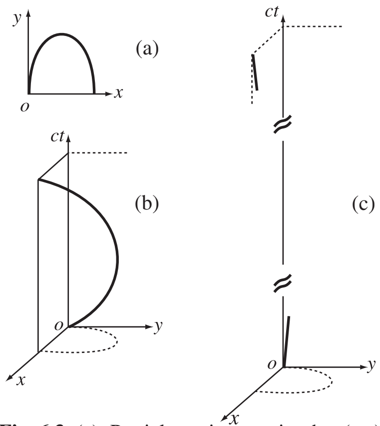

Fig. 6.3 (a) Particle trajectory in the (x,y) plane. (b) Particle worldline with projection onto the (x,y) plane as shown in (a). (c) Spacetime diagram with the time axis stretched a great distance.

Since the product $(dx^\sigma/d\lambda)(dx^\nu/d\lambda)$ in the second term is symmetric with respect to the interchange of indices σ and ν, only the symmetric part of its coefficient:

$$\frac{1}{2}\left(\frac{\partial g_{\mu\nu}}{\partial x^\sigma}+\frac{\partial g_{\mu\sigma}}{\partial x^\nu}\right)$$

can contribute. In this way the geodesic equation (6.7), after factoring out the common $g_{\mu\nu}$ coefficient, can be cast (after relabeling some repeated indices) into the form,

$$\frac{d^2x^\nu}{d\lambda^2}+\Gamma^\nu_{\sigma\rho}\frac{dx^\sigma}{d\lambda}\frac{dx^\rho}{d\lambda}=0, \tag{6.9}$$

where

$$g_{\mu\nu}\Gamma^\nu_{\sigma\rho}=\frac{1}{2}\left[\frac{\partial g_{\sigma\mu}}{\partial x^\rho}+\frac{\partial g_{\rho\mu}}{\partial x^\sigma}-\frac{\partial g_{\sigma\rho}}{\partial x^\mu}\right]. \tag{6.10}$$

$\Gamma^\nu_{\sigma\rho}$, defined as this particular combination of the first derivatives of the metric tensor, is called the **Christoffel symbol** (also known as the **affine connection**). The geometric significance of this quantity will be studied in Chapter 13. From now on (6.9) is the form of the geodesic equation that we shall use. To reiterate, the geodesic equation is the equation of motion in GR because it is the shortest curve in a warped spacetime. By this we mean that once the gravitational field is given, that is, spacetime functions $g_{\mu\nu}(x)$ and $\Gamma^\mu_{\nu\sigma}(x)$ are known, (6.9) tells us how a test particle will move in such a field: it will always follow the shortest and the straightest possible trajectory in this spacetime. A fuller justification of using the geodesic equation as the GR equation of motion will be given in Section 14.1.2.

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In Box 6.2 we demonstrate how the phenomenon of gravitational redshift follows directly from a curve space time description.

6.2.2 The Newtonian limit

Supporting our claim that the geodesic equation is the GR equation of motion, we shall now show that the geodesic equation (6.9) does reduce to the Newtonian equation of motion (6.6) in the **Newtonian limit** of a test particle moving with nonrelativistic velocity $v \ll c$ in a static and weak gravitational field.

- **Nonrelativistic speed** $(dx^i/dt) \ll c$: This inequality $dx^i \ll cdt$ implies that

$$\frac{dx^i}{d\lambda} \ll c\frac{dt}{d\lambda}\left(=\frac{dx^0}{d\lambda}\right). \tag{6.11}$$

  Keeping only the dominant term $(dx^0/d\lambda)(dx^0/d\lambda)$ in the double sum over indices λ and ρ of the geodesic equation (6.9), we have

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{00}\frac{dx^0}{d\lambda}\frac{dx^0}{d\lambda} = 0. \tag{6.12}$$

- **Static field** $(\partial g_{\mu\nu}/\partial x^0) = 0$: Because all time derivatives vanish, the Christoffel symbol of (6.10) takes the simpler form

$$g_{\nu\mu}\Gamma^\mu_{00} = -\frac{1}{2}\frac{\partial g_{00}}{\partial x^\nu}. \tag{6.13}$$

- **Weak field** $h_{\mu\nu} \ll 1$: We assume that the metric is not too different from the flat spacetime metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{6.14}$$

  where $h_{\mu\nu}(x)$ is a small correction field. Keeping in mind that flat space has a constant metric $\eta_{\mu\nu}$, we have $\partial g_{\mu\nu}/\partial x^\sigma = \partial h_{\mu\nu}/\partial x^\sigma$ and the Christoffel symbols are of order $h_{\mu\nu}$. To leading order, (6.13) is

$$\eta_{\nu\mu}\Gamma^\mu_{00} = -\frac{1}{2}\frac{\partial h_{00}}{\partial x^\nu} \tag{6.15}$$

  which, because $\eta_{\nu\mu}$ is diagonal, has for a static h_{00} the following components

$$-\Gamma^0_{00} = -\frac{1}{2}\frac{\partial h_{00}}{\partial x^0} = 0, \quad \text{and} \quad \Gamma^i_{00} = -\frac{1}{2}\frac{\partial h_{00}}{\partial x^i}. \tag{6.16}$$

  We can now evaluate (6.12) by using (6.16): the $\mu = 0$ equation leads to

$$\frac{dx^0}{d\lambda} = \text{constant}, \tag{6.17}$$

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and the three $\mu = i$ equations are

$$\frac{d^2x^i}{d\lambda^2} + \Gamma^i_{00}\frac{dx^0}{d\lambda}\frac{dx^0}{d\lambda} = \left(\frac{d^2x^i}{c^2dt^2} + \Gamma^i_{00}\right)\left(\frac{dx^0}{d\lambda}\right)^2 = 0, \tag{6.18}$$

where we have used (6.11) so that $(dx^i/d\lambda) = (dx^i/dx^0)(dx^0/d\lambda)$ and the condition of (6.17) to conclude $(d^2x^i/d\lambda^2) = (d^2x^i/dx^{0\,2})(dx^0/d\lambda)^2$. The above equation, together with (6.16), implies

$$\frac{d^2x^i}{c^2dt^2} - \frac{1}{2}\frac{\partial h_{00}}{\partial x^i} = 0, \tag{6.19}$$

which is to be compared with the Newtonian equation of motion (6.6). Thus $h_{00} = -2\Phi/c^2$ and using the definition of (6.14) we recover (6.3), first obtained heuristically in Section 6.1:

$$g_{00} = -\left(1 + \frac{2\Phi\,(x)}{c^2}\right). \tag{6.20}$$

We can indeed regard the metric tensor as the relativistic generalization of the gravitational potential. This expression also provides us with a criterion to characterize a field being weak as in (6.14):

$$[\ |h_{00}| \ll |\eta_{00}|\ ] \quad \Rightarrow \quad \left[\ \left|\Phi/c^2\right| \ll 1\ \right]. \tag{6.21}$$

Consider the gravitational potential at the earth's surface. It is equal to the gravitational acceleration times the earth's radius, $\Phi_\oplus = \mathfrak{g} \times R_\oplus = O(10^7 m^2/s^2)$, or $\Phi_\oplus/c^2 = O(10^{-10})$. Thus a weak field is any gravitational field being less than ten billion $\mathfrak{g}'s$.

Box 6.2 Gravitational re shift reestablished

Previously in Chapter 4 we have shown that the strong equivalence principle implied a gravitational redshift (in a static gravitational field) of light frequency ω

$$\frac{\Delta\omega}{\omega} = -\frac{\Delta\Phi}{c^2}. \tag{6.22}$$

From this result we heuristically deduced that, in the presence of a nonzero gravitational potential, the metric must deviate from the flat space value. That is, from the gravitation redshift we deduced a curved spacetime. Now we shall reestablish the redshift result, this time going the other way—starting with a curved spacetime, we shall deduce the redshift result.

In this chapter we have seen that Einstein's theory based on a curved spacetime has the result (6.20) in the Newtonian limit. This, as shown in (6.2), can be stated as a relation between the proper time τ and the coordinate time t as follows:

$$d\tau = \sqrt{-g_{00}}dt \quad \text{with} \qquad g_{00} = -\left(1 + 2\frac{\Phi}{c^2}\right). \tag{6.23}$$

Here we wish to see how the gravitational frequency shift result of (6.22) emerges in this curved spacetime description.

In Fig. 6.4, the two curvy lines are the light-like worldlines of two wavefronts emitted at an interval dt_{em} apart. They are curvy because in the presence of gravity the spacetime is curved. (In flat spacetime, they would be two straight 45^olines.) Because we are working with a static gravitational field (hence a time-independent spacetime curvature), this dt_{em} time interval between the two wavefronts is maintained throughout the trip until they are received. That is, these two wavefronts trace out two congruent worldlines. In particular the coordinate time separations at emission and reception are identical,

$$dt_{\text{em}} = dt_{\text{rec}}. \tag{6.24}$$

On the other hand, the frequency being inversely proportional to the proper time interval $\omega = 1/d\tau$, we can then use (6.23) and (6.24) to derive:

$$\frac{\omega_{\text{rec}}}{\omega_{\text{em}}} = \frac{d\tau_{\text{em}}}{d\tau_{\text{rec}}} = \frac{\sqrt{-(g_{00})_{\text{em}}}\,dt_{\text{em}}}{\sqrt{-(g_{00})_{\text{rec}}}\,dt_{\text{rec}}} = \left(\frac{1 + 2(\Phi_{\text{em}}/c^2)}{1 + 2(\Phi_{\text{rec}}/c^2)}\right)^{1/2}$$

$$= 1 + \frac{\Phi_{\text{em}} - \Phi_{\text{rec}}}{c^2} + O\left(\Phi^2/c^4\right), \tag{6.25}$$

which is the claimed result of (6.22):

$$\frac{\omega_{\text{rec}} - \omega_{\text{em}}}{\omega_{\text{em}}} = \frac{\Phi_{\text{em}} - \Phi_{\text{rec}}}{c^2}. \tag{6.26}$$

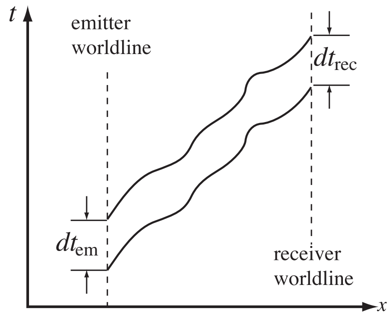

Fig. 6.4 Worldlines for two light wavefronts propagating from emitter to receiver in a static curved spacetime.

6.3 The curvature of spacetime

We have already discussed in Chapter 5 (see especially Section 5.2.2) that in a curved space each small region can be approximated by a flat space, that is, locally one can always perform a coordinate transformation so the new metric is approximately a flat-space metric. This coordinate dependence of the metric shows that the metric value cannot represent the core feature of a curved space. However, as shown in Section 5.3 (and further discussion in Section 13.3), there exits a mathematical quantity involving the second derivative of the metric, called the curvature, which does represent the essence of a curved space: the space is curved if and only if the curvature is nonzero; and, also, the deviations from Euclidean relations are always proportional to the curvature.

If the warped spacetime is the gravitational field, what then is its curvature? What is the physical manifestation of this curvature? How does it enter in the GR gravitational field equation?

6.3.1 Tidal force as the curvature of spacetime

The equivalence principle states that in a freely falling reference frame the physics is the same as that in an inertial frame with no gravity. SR applies and the metric is given by the Minkowski metric $\eta_{\mu\nu}$. As shown in the flatness theorem (Section 5.2.2), this approximation of $g_{\mu\nu}$ by $\eta_{\mu\nu}$ can be done **only** locally, that is, in an appropriately small region. Gravitational effects can always be detected in a **finite-sized** free-fall frame as the gravitational field is never strictly uniform in reality; the second derivatives of the metric come into play.[4]

Consider the lunar gravitational attraction exerted on the earth. While the earth is in free fall toward the moon (and vice versa), there is still a detectable lunar gravitational effect on the earth. This is so because different points on the earth will feel slightly different gravitational pulls by the moon, as depicted in Fig. 6.5(a). The center-of-mass (CM) force causes the earth to "fall towards the moon" so that this CM gravitational effect is "cancelled out" in this freely falling terrestrial frame. After subtracting out this CM force, the remanent forces on the earth, as shown in Fig. 6.5(b), are stretching in the longitudinal direction and compression in the transverse direction. They are just the familiar tidal forces.[5] That is, in the freely falling frame, the CM gravitational effect is transformed away, but, there are still the remnant tidal forces. They reflect the **differences** of the gravitational effects on neighboring points, and are thus proportional to the derivative of the gravitational field.

We can illustrate this point by the following observation. With r_{s} and r_{m} being the distances from the earth to the sun and moon, respectively,

[4]The flatness theorem states that in the local inertial frame $(\bar{x}^{\mu})$ the new metric is, according to (5.31), approximately flat: $\bar{g}_{\mu\nu}(\bar{x}) = \eta_{\mu\nu} + \gamma_{\mu\nu\lambda\rho}(0)\bar{x}^{\lambda}\bar{x}^{\rho} + \cdots$ with $\partial\bar{g}_{\mu\nu}/\partial\bar{x}^{\lambda} = 0$.

[5]The ocean is pulled away in opposite directions giving rise to two tidal bulges. This explains why, as the earth rotates, there are two high tides in a day. This is a simplified account as there are also other effects (e.g. the solar tidal forces).

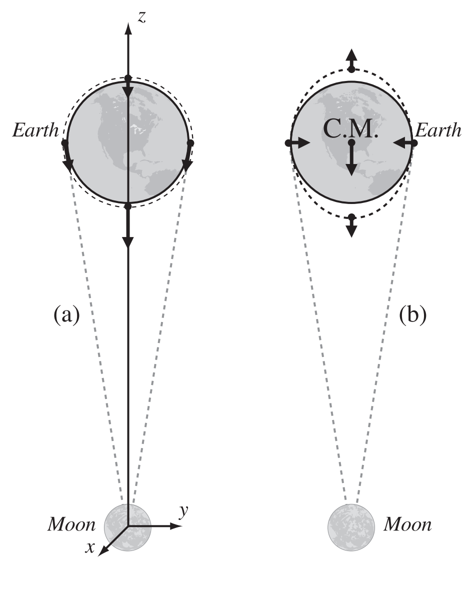

Fig. 6.5 Variations of the gravitational field as tidal forces. (a) Lunar gravitational forces on four representative points on the earth. (b) After taking out the center of mass (CM) motion, the relative forces on the earth are the tidal forces giving rise to longitudinal stretching and transverse compression.

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we have

$$\left[\mathfrak{g}_{\mathrm{s}}=\frac{G_{\mathrm{N}}M_{\odot}}{r_{\mathrm{s}}^{2}}\right]>\left[\mathfrak{g}_{\mathrm{m}}=\frac{G_{\mathrm{N}}M_{\mathrm{m}}}{r_{\mathrm{m}}^{2}}\right], \tag{6.27}$$

showing that the gravitational attraction of the earth by the sun is much larger than that by the moon. On the other hand, because the tidal force is given by the derivative of the field strength

$$\left[\frac{\partial}{\partial r}\frac{G_{\mathrm{N}}M}{r^{2}}\right]\propto\left[\frac{G_{\mathrm{N}}M}{r^{3}}\right], \tag{6.28}$$

and because $r_{\mathrm{s}} \gg r_{\mathrm{m}}$, the lunar tidal forces nevertheless end up being stronger than the solar ones:

$$\left[\mathfrak{T}_{\mathrm{s}}=\frac{G_{\mathrm{N}}M_{\odot}}{r_{\mathrm{s}}^{3}}\right]<\left[\mathfrak{T}_{\mathrm{m}}=\frac{G_{\mathrm{N}}M_{\mathrm{m}}}{r_{\mathrm{m}}^{3}}\right]. \tag{6.29}$$

Since tidal forces cannot be coordinate-transformed away, they should be regarded as the essence of gravitation. They are the variations of the gravitational field, hence the second derivatives of the gravitational potential. From the discussion in this chapter showing that the relativistic gravitational potential being the metric, and that second derivative of the metric being the curvature, we see that Einstein gives gravity a direct geometric interpretation by identifying these tidal forces with the curvature of spacetime. A discussion of tidal forces in terms of the Newtonian deviation equation is given in Box 6.3.

Box 6.3 The equation of Newtonian deviation and its GR generalization

Here we provide a more quantitative description of the gravitational tidal force in the Newtonian framework, which will suggest an analogous GR approach to be followed in Chapter 14.

As the above discussion indicates, the tidal effect concerns the relative motion of particles in a nonuniform gravitational field. Let us consider two particles: one has the trajectory $\vec{x}(t)$ and another has $\vec{x}(t)+\vec{s}(t)$. That is, the locations of these two particles measured at the same time has a coordinate difference of $\vec{s}(t)$. The respective equations of motion ($i=1,2,3$) obeyed by these two particles are:

$$\frac{d^{2}x^{i}}{dt^{2}}=-\frac{\partial\Phi(x)}{\partial x^{i}} \quad \text{and} \quad \frac{d^{2}x^{i}}{dt^{2}}+\frac{d^{2}s^{i}}{dt^{2}}=-\frac{\partial\Phi(x+s)}{\partial x^{i}}. \tag{6.30}$$

(*cont.*)

Box 6.3 (*Continued*)

Consider the case where the separation distance $s^i(t)$ is small and we can approximate the gravitational potential $\Phi(x+s)$ by a Taylor expansion

$$\Phi(x+s) = \Phi(x) + \frac{\partial \Phi}{\partial x^j} s^j + \cdots \tag{6.31}$$

From the difference of the two equations in (6.30), we obtain the **Newtonian deviation equation** that describes the separation between two particle trajectories in a gravitational field

$$\frac{d^2 s^i}{dt^2} = -\frac{\partial^2 \Phi}{\partial x^i \partial x^j} s^j. \tag{6.32}$$

Thus the relative acceleration per unit separation $(d^2s^i/dt^2)/s^j$ is given by a tensor having the second derivatives of the gravitational potential (i.e. the tidal force components) as its elements.

We now apply (6.32) to the case of a spherical gravitational source (e.g. the gravity due to the moon on earth as shown in Fig. 6.5), $\Phi(x) = -G_{\mathrm{N}}M/r$, where the radial distance is related to the rectangular coordinates by $r = (x^2+y^2+z^2)^{1/2}$. Since $\partial r/\partial x^i = x^i/r$ we have

$$\frac{\partial^2 \Phi}{\partial x^i \partial x^j} = \frac{G_N M}{r^3}\left(\delta_{ij} - \frac{3x^i x^j}{r^2}\right). \tag{6.33}$$

Consider the case of the"first particle" being located along the z axis $x^i = (0, 0, r)$. The Newtonian deviation equation (6.32) for the displacement of the"second particle", with the second derivative tensor given by (6.33), now takes on the form

$$\frac{d^2}{dt^2}\begin{pmatrix} s_x \\ s_y \\ s_z \end{pmatrix} = \frac{-G_N M}{r^3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} s_x \\ s_y \\ s_z \end{pmatrix}. \tag{6.34}$$

We see that there is an attractive tidal force between the two particles in the transverse direction $\mathfrak{T}_{x,y} = -G_N M r^{-3} s_{x,y}$ which leads to compression; and a tidal repulsion $\mathfrak{T}_z = +2G_N M r^{-3} s_z$, leading to stretching, in the longitudinal (i.e. radial) direction.

In GR, we shall follow a similar approach (see Problems 14.4 and 14.5): the two equations of motion (6.30) will be replaced by the corresponding geodesic equations; their difference, after a Taylor expansion, leads to the **equation of geodesic deviation**, which is entirely similar[6] to (6.32). Since the metric function is the relativistic potential, the second derivative tensor turns into the curvature tensor of the spacetime (the Riemann curvature tensor). In this geometric language we see that the cause of the deviation from a flat spacetime worldline is attributed to the curvature. Recall the example discussed previously in Section 6.1.2; see especially Fig. 6.2.

[6] We are not quite ready to derive this GR equation as one still needs to learn how to perform differentiations in a curved space (see Chapter 13).

6.3.2 The GR field equation described

We now discuss how the curvature, identified as the tidal forces, enters directly in the field equations of relativistic gravitational theory.

The field equation relates the source distribution to the resultant field; given the source distribution, we can use the field equation to find the field everywhere. For the Newtonian equation (6.5),

$$\nabla^2 \Phi = 4\pi G_{\mathrm{N}} \rho, \tag{6.36}$$

we have the second derivative of the gravitational potential $\nabla^2 \Phi$ being directly proportional to the mass density ρ. What is the relativistic generalization of this equation?

1. For the right-hand side (RHS) of the field equation, from the viewpoint of relativity, mass being just a form of energy (the rest energy) and, furthermore, energy and momentum being equivalent, they can be transformed into each other when viewed by different observers (see (3.37)), the mass density ρ of (6.5) is generalized in relativity to an object called the "**energy–momentum tensor**" $T_{\mu\nu}$. The 16 elements include T_{00}, being the energy density ρc^2, three elements T_{0i} begin the momentum densities, and the remaining 12 elements representing the fluxes associated with the energy and momentum densities—they describe the flow of energy and momentum components. There are actually only 10 independent elements, because it is a symmetric tensor $T_{\mu\nu} = T_{\nu\mu}$. A more detailed discussion of the energy–momentum tensor will be presented in Section 12.3.
2. For the second derivative of the potential on the left-hand side (LHS) of the field equation, we have already seen that the relativistic gravitational potential is the metric $g_{\mu\nu}$ and the curvature in (5.35) is the second derivative of the metric. And, as we shall find in Chapter 13, for higher dimensional spaces, the Gaussian curvature K of Chapter 5 is generalized to the **Riemann curvature tensor**. A particular (contracted) version of the curvature tensor is the **Einstein tensor** $G_{\mu\nu}$, having mathematical properties that match those of the energy–momentum tensor $T_{\mu\nu}$.

This suggests the possible relativistic generalization of the gravitational field equation as having the basic structure of (6.36): the RHS being the energy–momentum tensor, and the LHS being the Einstein tensor involving the second derivative of the metric:

Newton equation	$\nabla^2 \Phi \propto \rho$
Einstein equation	$G_{\mu\nu} \propto T_{\mu\nu}$

In this way, we obtain in Section 14.2 the GR field equation, the **Einstein equation,** in the form of

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \tag{6.37}$$

where κ is a proportionality constant. As we shall show in Section 14.2.2 the nonrelativistic limit of this equation is just Newton's equation (6.36) when we make the identification of

$$\kappa = -\frac{8\pi G_{\mathrm{N}}}{c^4}. \tag{6.38}$$

Since the curvature has a different measurement unit from that for the energy–momentum density,[7] the proportional constant κ, hence Newton's constant G_{N}, should be interpreted as a **conversion factor**. Just as the speed of light c is the conversion factor between space and time that is fundamental to the special relativistic symmetry of space and time (see Section 3.4), one way of viewing the significance of Newton's constant is that it is the conversion factor[8] fundamental for a geometric description of gravity by GR; it connects the spacetime curvature to the gravitational source of energy and momentum, as in Einstein's equation:

[7] *NB:* the elements of the energy–momentum tensor have the dimension of energy density, while the dimension of curvature is $(\text{length})^{-2}$.

[8] We also note that Planck's constant is a conversion factor fundamental for quantum theory, which connects the wave and particle descriptions.

$$\begin{pmatrix} \text{curvature} \\ \text{of spacetime} \end{pmatrix} = (\text{Newton's constant}) \times \begin{pmatrix} \text{energy–momentum} \\ \text{density.} \end{pmatrix}.$$

When worked out in Chapter 14, we shall see that (6.37) represents 10 coupled partial differential equations. Their solution is the metric function $g_{\mu\nu}(x)$, fixing the geometry of spacetime. We emphasize once more that in GR, spacetime is no longer a passive background against which physical events take place. Rather, it is a dynamical entity as it responds to the ever-changing matter/energy distribution in the world. But in more conventional field theory language, the metric is the gravitational field, and energy–momentum density is the "gravity charge."

For the rest of Parts II and III (Chapters 7–11), this Einstein field equation will not be discussed further. Rather, we shall concentrate on investigating its solutions, showing how a curved spacetime description of the gravitational field, that is, knowing the metric $g_{\mu\nu}(x)$, brings about many interesting physical consequences: from bending of light rays, and black holes, to cosmology.

GR and the structure of spacetime

Einstein was motivated to build a new theory of gravity that is compatible with the principle of relativity and has the principle of equivalence fundamentally built into the theory. Remarkably all this could be implemented simply by allowing for a curved spacetime instead of the flat one for SR. In GR gravity is not a force, but a change of structure of spacetime that allows inertial observers to accelerate with respect to each other. The warped spacetime can account for all gravitational phenomena. The Einstein equation shows how the energy–momentum density can change the curvature of spacetime. In this way, GR brings about a radical change in the way we describe physical events: in SR physical events are depicted in a fixed spacetime, while in GR the spacetime itself is a dynamical entity, ever-evolving in response to the presence of energy and momentum.

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Box 6.4 Einstein's three motivations: An update

In Chapter 1 we discussed Einstein's motivations for creating general relativity. Now we can see how these issues are resolved in the curved spacetime formulation of a relativistic theory of gravitation.

1. **SR is not compatible with gravity** In the GR formulation, we see that SR is valid only in the local inertial frames in which gravity is transformed away.
2. **A deeper understanding of $m_I = m_G$** The weak EP is generalized to strong EP. The various consequences of the equivalence principle led Einstein to the idea of a curved spacetime as the relativistic gravitational field. At the fundamental level there is no difference between gravity and the "fictitious forces" associated with accelerated frames. Noninertial frames of reference in Newtonian physics are identified in Einstein's theory with the presence of gravity. The GR theory, symmetric with respect to general coordinate transformations (including accelerated coordinates), and the relativistic field theory of gravitation must be one and the same.[9]The equivalence principle is built right into the curved spacetime description of gravitation because any curved space is locally flat.
3. **"Space is not a thing"** The GR equations are covariant under the most general (position-dependent) coordinate transformations.[10]GR physics is valid in any coordinate frame. Furthermore, spacetime (metric) is the solution to the Einstein equation. It has no independent existence except expressing the relation among physical processes in the world.

[9] GR theory respects the symmetry with respect to general coordinate transformation, including the accelerated coordinates. EP teaches us that these accelerated frames are equivalent to the inertial frames with gravity. Hence general relativistic theory must necessarily be a theory of gravity.

[10]See further discussion as the"principle of general covariance" in Section 14.1.

Review questions

1. What does one mean by a"geometric theory of physics?" Use the distance measurements on the surface of a globe to illustrate your answer.
2. How can the phenomenon of gravitational time dilation be phrased in geometric terms? Use this discussion to support the suggestion that the spacetime metric can be regarded as the relativistic gravitational potential.
3. Give the simple example of a rotating cylinder to illustrate how the EP physics implies a non-Euclidean geometric relation.
4. What significant conclusion did Einstein draw from the analogy between the fact that a curved space is locally flat and that gravity can be transformed away locally?
5. How does GR imply a conception of space and time as reflecting merely the relationship between physical events rather than a stage onto which physical events take place.
6. Give a heuristic argument for the GR equation of motion to be the geodesic equation.
7. What is the Newtonian limit? In this limit, what relation can one infer between the Newtonian gravitational potential and a metric tensor component of the spacetime. Use this relation to derive the gravitational redshift.

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8. What are tidal forces? How are they related to the gravitational potential? Explain why the solar tidal forces are smaller than lunar tidal forces, even though the gravitational attraction of the earth by the sun is stronger than that by the moon. Explain how in general relativity the tidal forces are identified with the curvature of spacetime.
9. Give a qualitative description of the GR field equation. Explain in what sense we can regard Newton's constant as a basic"conversion factor" in general relativity. Can you name two other conversion factors in physics that are respectively basic to special relativity and to quantum theory?
10. How are Einstein's three motivations for creating GR resolved in the final formulation of the geometric theory of gravity?

Problems

6.1 **The metric element g_{00}** From the definitions of metric and proper time, derive the relation between proper time and coordinate time;

$$d\tau = \sqrt{-g_{00}}dt.$$

6.2 **Spatial distance and spacetime metric** Einstein suggested the following definition of the spatial distance dl between two neighboring points (A, B) with a coordinate difference of dx^i (where $i = 1, 2, 3$). A light pulse is sent to B and reflected back to A. If the elapsed proper time (according to A) is $d\tau_A$, then $dl \equiv cd\tau_A/2$. The square of the spatial distance should also be quadratic in dx^i

$$dl^2 = \gamma_{ij}dx^i dx^j.$$

How is this spatial metric γ_{ij} related to the spacetime metric $g_{\mu\nu}$ as defined by $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ (where $\mu = 0, 1, 2, 3$)? Is it just $\gamma_{ij} = g_{ij}$? *Suggestion*: Solve the $ds^2 = 0$ equation in terms of the various elements of $g_{\mu\nu}$, for the coordinate time required for the light signal to make the round trip. See Landau and Lifshitz (1975, §84).

6.3 **Non-Euclidean geometry of a rotating cylinder** In Section 6.1.1 we used the example of a rotating cylinder to motivate the need for non-Euclidean geometry. Use the formalism derived in Problem 6.2 to work out the spatial distance, showing this violation of the Euclidean relation between radius and circumference.

6.4 **Geodesic equation in a rotating coordinate** Knowing the metric for a rotating coordinate from Problem 6.3, work out the corresponding Christoffel symbols and geodesic equation. This can be taken as the relativistic version of the centrifugal force.

6.5 **The geodesic equation and light deflection** Use the geodesic equation, rather than Huygens' principle, to derive the expression of gravitational angular deflection given by (4.44) and (4.45), if the only warped metric element is $g_{00} = -1 - 2\Phi(x)/c^2$. One approach is to note (see Fig. 4.6) that the infinitesimal angular deflection of a photon with momentum $\vec{p} = p\hat{x}$ is related to momentum change by $d\phi = dp_y/p$. In turn we can always choose a curve parameter τ for the light geodesic so that the photon momentum equals the derivative of displacement with respect to such a parameter $p^\mu = dx^\mu/d\tau$ with $\mu = 0, 1, 2, 3$. For the photon momentum 4-vector in the $\hat{x}$ direction, we have $p^\mu = (p, p, 0, 0)$ (see Section 3.2.2). The deflection can be calculated from the geodesic equation by its determination of $dx^\mu/d\tau$, hence p^μ.

6.6 **Symmetry property of the Christoffel symbols** From the definition of (6.10), check explicitly that

$$\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}.$$

6.7 **The matrix for tidal forces is traceless** One notes that the matrix in (6.34) is traceless (vanishing sum of the diagonal elements). Why should this be so?

6.8 **G_N as a conversion factor** From Newton's theory we know that Newton's constant has the dimension of energylengthmass^{-2}. With such a G_N in the proportional constant (6.38) of the Einstein equation (6.37), check that it yields the correct dimension for the curvature on the LHS of the Einstein equation.