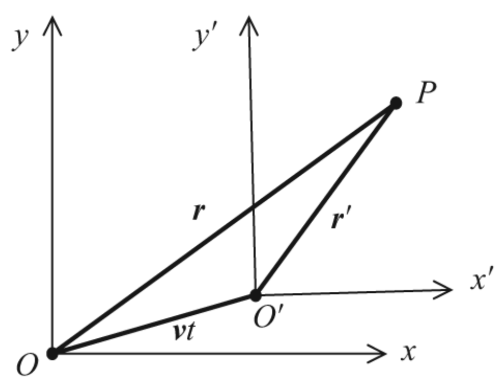
Ch 1, 2, 3 solutions

A College Course on Relativity and Cosmology by Ta-Pei Cheng
Appendix E

(1.1) Graphic representation of Galilean transformation: Draw a (2D) vector diagram showing the Galilean relation of $\mathbf{r}' = \mathbf{r} - \mathbf{v}t$ for a general relative relative velocity, instead of the restricted case of $\mathbf{v} = v\hat{\mathbf{x}}$ as shown in Fig 1.2.

1.1A Graphic representation of Galilean transformation for a general $\mathbf{v}$.



Exercise 1.1 Two coordinate systems in motion with a general relative velocity.

(1.2) Galilean covariance of Newton's law: Demonstrate that the law of universal gravitational attraction,

$$\mathbf{F} = G_{\mathrm{N}} \frac{m_A m_B}{r_{AB}^3} \mathbf{r}_{AB}, \tag{E.1}$$

is unchanged under Galilean transformation. In the above equation we have two mass points m_A and m_B, located at positions $\mathbf{r}_A$ and $\mathbf{r}_B$, separated by $\mathbf{r}_{AB} = \mathbf{r}_A - \mathbf{r}_B$, pointing from B to A. The force on either mass due to the other is related to the acceleration by Newton's Second Law, $\mathbf{F} = -m_A \mathbf{a}_A = m_B \mathbf{a}_B$.

1.2A Because the $\mathbf{v}t$ term is the same for both $\mathbf{r}'_A$ and $\mathbf{r}'_B$, we have an invariant separation: $\mathbf{r}'_{AB} = \mathbf{r}'_A - \mathbf{r}'_B = \mathbf{r}_A - \mathbf{r}_B = \mathbf{r}_{AB}$. Thus the RHS is invariant under the transformation. As for the LHS $\mathbf{F} = m\mathbf{a}$, time-differentiating the position relation (using $t' = t$), we have $d\mathbf{r}/dt' = d\mathbf{r}/dt - \mathbf{v}$; differentiating one more time, we see that acceleration is unchanged: $d^2\mathbf{r}'/dt'^2 = d^2\mathbf{r}/dt^2$ because the relative velocity is a constant, $d\mathbf{v}/dt = 0$. Consequently every term in Newton's gravity law are invariant under a Galilean transformation. Clearly the same equation holds in the new inertial frame of reference.

(1.3) Galilean covariance of Newtonian momentum conservation: Consider the two-particle collision $A + B \longrightarrow C + D$. Demonstrate that if momentum conservation holds in one frame O,

$$m_A \mathbf{u}_A + m_B \mathbf{u}_B = m_C \mathbf{u}_C + m_D \mathbf{u}_D, \tag{E.2}$$

it also holds in another frame O' in relative motion $(\mathbf{v})$, provided the total mass is also conserved: $m_A + m_B = m_C + m_D$.

1.3A We need to check the validity of

$$(m_A \mathbf{u}'_A + m_B \mathbf{u}'_B) - (m_C \mathbf{u}'_C + m_D \mathbf{u}'_D) = 0. \tag{E.3}$$

Given the Galilean velocity addition rule of $\mathbf{u}'_i = \mathbf{u}_i - \mathbf{v}$, with $i = A, B, C, D$, the above equation becomes

$$(m_A \mathbf{u}_A + m_B \mathbf{u}_B) - (m_C \mathbf{u}_C + m_D \mathbf{u}_D) = \mathbf{v}\left[(m_A + m_B) - (m_C + m_D)\right].$$

This relation holds because the LHS vanishes by (E.2) and RHS vanishes by mass conservation.

(1.4) The SR velocity addition rule: For motion in one spatial dimension (x only), the space and time coordinates transform according to the Lorentz transformation of (1.16). From its differential form,

$$dx' = \gamma\,(dx - v dt)\,, \qquad dt' = \gamma\left(dt - \frac{v}{c^2} dx\right), \tag{E.4}$$

prove this new SR velocity addition rule, replacing the familiar relation of (1.10) with

$$u' = \frac{u-v}{1-\frac{uv}{c^2}}, \tag{E.5}$$

where $u = dx/dt$ and $u' = dx'/dt'$ are the velocities of a particle as measured in two reference frames in relative motion (v).

1.4A The differential form of the Lorentz transformation being

$$dx' = \gamma\,(dx - vdt)\,, \qquad dt' = \gamma\left(dt - \frac{v}{c^2}dx\right), \tag{E.6}$$

the ratio of these two equations is

$$\frac{dx'}{dt'} = \frac{dx - vdt}{dt - \frac{v}{c^2}dx}. \tag{E.7}$$

Since $u = dx/dt$ and $u' = dx'/dt'$, this equation of the ratio is the new velocity addition rule$u' = (u-v)\,/\left(1 - uv/c^2\right)$.

(2.1) Illustrating the relativity of equilocality: While the notion of simultaneity's relativity may appear strange to us, we are all familiar with its analog in spatial coordinates — the relativity of equilocation. Two events that take place at the same location (but at different times) will be seen by a moving observer to have happened at different positions. This is a straightforward consequence of Galilean transformation. Use the setup as shown in Fig. 2.2 (*i.e.*, light emissions at a fixed location on the railcar) to illustrate this phenomenon of relativity of equilocality.

2.1A We can illustrate this by considering the emission of two sequential light pulses from a fixed location on the railcar (e.g., the front end). While these two events are clearly equilocal to an observer on the railcar, they are seen as taking place at two different locations when viewed by the passing observer standing on the rail platform.

(2.2) Calculating the nonsynchronicity of two events: In Fig. 2.2 the two events (light pulses arriving at the front and back ends of the railcar) are viewed as simultaneous in the O' frame: $t'_1 = t'_2$. (a) Work out the nonsynchronicity, $\triangle t = t_2 - t_1$, of these two events as viewed by an observer on the ground as the train of length L in the platform observer's (O) frame passes by with speed v. (b) Show that simultaneity would be absolute, $\triangle t = \triangle t'$, had we followed the classical velocity addition rule (1.10), so that light signals would propagate forward with speed $c+v$ and backward with $c-v$.

2.2A To the observer on the ground, light travels a shorter distance to reach the back-end of the rail car and an extra distance to the front-end. (a)

Because of the constancy of the light speed, their arrival times will be different:

$$t_1 = \frac{\frac{L}{2} - vt_1}{c}, \qquad \text{or} \quad t_1 = \frac{\frac{L}{2}}{c+v}, \tag{E.8}$$

and

$$t_2 = \frac{\frac{L}{2} + vt_2}{c}, \qquad \text{or} \quad t_2 = \frac{\frac{L}{2}}{c-v}. \tag{E.9}$$

The amount of nonsynchronicity is then

$$\triangle t = t_2 - t_1 = \frac{L}{2} \frac{2v}{c^2 - v^2} = \gamma^2 \frac{v}{c^2} L. \tag{E.10}$$

(b) Had the light propagates with $c \pm v$ speeds, the two events would be seen as simultaneous in the O frame, just as so in the O' frame:

$$t_1 = \frac{\frac{L}{2} - vt_1}{c - v}, \qquad \text{or} \quad t_1 = \frac{L}{2c} = t_2. \tag{E.11}$$

(2.3) Lorentz transformation for a general relative velocity: The Lorentz transformation given in (2.12) is the special case in which the relative velocity $\mathbf{v}$ of the two frames is along the direction of the x axis. Namely, the coordinate system is chosen such that the x axis is parallel to the relative velocity $\mathbf{v}$. For the case where $\mathbf{v}$ is of a general direction, show that the position transformation may be written as

$$\mathbf{r}' = \mathbf{r} + (\gamma - 1) \frac{\mathbf{r} \cdot \mathbf{v}}{v^2} \mathbf{v} - \gamma \mathbf{v} t. \tag{E.12}$$

2.3A The Lorentz transformation (2.12) for a general velocity written in terms of the parallel and perpendicular position components is as follows:

$$\begin{aligned} r'_{\|} &= \gamma \left(r_{\|} - vt \right), \qquad \mathbf{r}'_{\perp} = \mathbf{r}_{\perp} \\ t' &= \gamma \left(t - \frac{\mathbf{v} \cdot \mathbf{r}}{c^2} \right). \end{aligned} \tag{E.13}$$

Since parallel components are along the direction of the unit vector $\mathbf{v}/v$ and perpendicular components are $\mathbf{r}_{\perp} = \mathbf{r} - \mathbf{r}_{\|}$ and are not changed by the transformation: $\mathbf{r}' = \mathbf{r}'_{\|} + \mathbf{r}'_{\perp} = \mathbf{r}'_{\|} + \mathbf{r}_{\perp}$, substituting in the above transformation we obtain

$$\begin{aligned} \mathbf{r}' &= r'_{\|} \frac{\mathbf{v}}{v} + \left(\mathbf{r} - r_{\|} \frac{\mathbf{v}}{v} \right) = \mathbf{r} + \left(r'_{\|} - r_{\|} \right) \frac{\mathbf{v}}{v} \\ &= \mathbf{r} + \left[(\gamma - 1) r_{\|} - \gamma vt \right] \frac{\mathbf{v}}{v} = \mathbf{r} + (\gamma - 1) \frac{\mathbf{v} \cdot \mathbf{r}}{v^2} \mathbf{v} - \gamma \mathbf{v} t. \end{aligned} \tag{E.14}$$

(2,4) The transformation of coordinate derivatives via the chain rule: Given the transformation for the space and time coordinates, find the

Lorentz transformation for the coordinate derivatives (2.18) by the chain rule of differentiation:

$$\begin{aligned}\frac{\partial}{\partial x'} &= \frac{\partial}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial}{\partial t}\frac{\partial t}{\partial x'} \\ \frac{\partial}{\partial t'} &= \frac{\partial}{\partial x}\frac{\partial x}{\partial t'} + \frac{\partial}{\partial t}\frac{\partial t}{\partial t'}.\end{aligned} \tag{E.15}$$

The partial derivatives ($\partial x/\partial x'$, etc.) can be read off from the differential form of (2.14) by interpreting it also as a train rule equation:

$$\begin{aligned}dx &= \frac{\partial x}{\partial x'}dx' + \frac{\partial x}{\partial t'}dt' \\ dt &= \frac{\partial t}{\partial x'}dx' + \frac{\partial t}{\partial t'}dt'.\end{aligned} \tag{E.16}$$

Namely, the transformation is a matrix of partial derivatives

$$\begin{pmatrix} \partial'_x \\ \partial'_0 \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial x'} & \frac{\partial x^0}{\partial x'} \\ \frac{\partial x}{\partial x'^0} & \frac{\partial x^0}{\partial x'^0} \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_0 \end{pmatrix}. \tag{E.17}$$

2.4A The (inverse) Lorentz transformation for infinitesimal coordinates being

$$\begin{aligned}dx &= \gamma\left(dx' + v dt'\right) \\ dt &= \gamma\left(dt + \frac{v}{c^2}dx'\right),\end{aligned} \tag{E.18}$$

we can identify the transformation matrix elements with partial derivatives in (E.16) as

$$\frac{\partial x}{\partial x'} = \gamma, \qquad \frac{\partial x}{\partial t'} = \gamma v, \qquad \frac{\partial t}{\partial x'} = \gamma\frac{v}{c^2}, \qquad \frac{\partial t}{\partial t'} = \gamma. \tag{E.19}$$

Substitute them into (E.15) we obtained the transformation for coordinate derivatives

$$\begin{aligned}\frac{\partial}{\partial x'} &= \gamma\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right) \\ \frac{\partial}{\partial t'} &= \gamma\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right),\end{aligned} \tag{E.20}$$

which is just the relation displayed in (2.18).

(2.5) **Use a light-pulse clock to show length contraction:** In Fig. 2.3 we used a light-pulse clock to demonstrate the phenomenon of time dilation. This same clock can be used to demonstrate length contraction. Suppose the clock moves parallel, rather than perpendicular, to the bouncing light pulses. The length of the clock L' can be measured in the rest frame of the clock through the time interval $\Delta t'$ that it takes a light pulse to make the trip across the length of the clock and back: $c\,(\Delta t_1' + \Delta t_2') = c\Delta t' = 2L'$. In the moving reference frame the length and time are measured to be L and Δt. A naive application of the time-dilation formula of $\Delta t = \gamma \Delta t'$ would suggest the incorrect effect of length elongation, $L = \gamma L'$. Demonstrate that a careful consideration of the light clock's operation in this setup does lead to the expected result of $L = L'/\gamma$.

2.5A Let the time interval for the light to travel from the left-end to the right-end be Δt_1 and when in the opposite direction, Δt_2. In the rest frame of the clock, we clearly have $\Delta t_1' = \Delta t_2'$. The length measured by the observer at rest with respect to the clock is $2L' = c\Delta t'$ with $\Delta t' = \Delta t_1' + \Delta t_2'$. When the clock is moving along the direction parallel to the light pulse, say, from left to right, the propagation time intervals in the two directions are not the same (as the light needs to travel further to reach the right-end, and travel less when bouncing back to the left-end):

$$c\Delta t_1 = L + v\Delta t_1, \qquad c\Delta t_2 = L - v\Delta t_2. \tag{E.21}$$

The round trip time becomes

$$\Delta t_1 + \Delta t_2 = \frac{L}{c-v} + \frac{L}{c+v} = \frac{2L}{c}\gamma^2. \tag{E.22}$$

For the LHS we can apply the time dilation formula $\Delta t = \gamma \Delta t'$ and then convert it to the rest frame length L' via $c\Delta t' = 2L'$,

$$\Delta t_1 + \Delta t_2 = \gamma\,(\Delta t_1' + \Delta t_2') = \gamma \frac{2L'}{c}. \tag{E.23}$$

A comparison of the RHSs of (E.22) and (E.23) leads then to the length contraction result of $L = L'/\gamma$.

(2.6) **Lorentz contraction of a moving sphere:** A sphere of radius R is depicted as $x^2 + y^2 + z^2 = R^2$. A moving observer O' with speed v will see this sphere having the shape of ellipsoid:

$$\frac{x'^2}{X^2} + \frac{y'^2}{Y^2} + \frac{z'^2}{Z^2} = 1 \qquad \text{with a volume } V' = \frac{4\pi}{3} XYZ. \tag{E.24}$$

How is this ellipsoidal volume related to the original spherical volume?

2.6A The equation for the sphere may be written as

$$\frac{x^2}{R^2} + \frac{y^2}{R^2} + \frac{z^2}{R^2} = 1 \tag{E.25}$$

The moving frame (x', y', z') coordinates are related to we have $x' = x/\gamma,\ y' = y,\ z' = z$, and the spherical equation becomes

$$\frac{\gamma^2 x'^2}{R^2} + \frac{y'^2}{R^2} + \frac{z'^2}{R^2} = 1 \tag{E.26}$$

which is an equation of ellipsoid

$$\frac{x'^2}{X^2} + \frac{y'^2}{Y^2} + \frac{z'^2}{Z^2} = 1 \tag{E.27}$$

with $X = R/\gamma,\ Y = Z = R$. The volume of an ellipsoid is $\frac{4\pi}{3}XYZ = \left(\frac{4\pi}{3}R^3\right)/\gamma$, just the spherical volume contracted by a Lorentz factor.

(2.7) Constancy of light velocity in a general direction: In Chapter 1 [cf (1.18)] we showed that if light travels in the same direction as the relative velocity of two observers, each observer sees the light propagate with the same speed, $u' = u = c$. Prove this result for a light pulse moving in an arbitrary direction. In principle one can follow the same procedure and work out the three components $u_i' = dx_i'/dt'$ from the Lorentz transformation of the infinitesimal intervals and then show that for such a light pulse the magnitude of $\mathbf{u}$ is invariant. However this approach involves a rather laborious calculation. Here you are asked to follow a much more efficient route by using the invariant interval ds^2, defined in (2.28), for your proof.

2.7A The invariance of the infinitesimal interval $ds'^2 = ds^2$ certainly applies to the particular case of light: If we have $ds^2 = 0$, we must also have $ds'^2 = 0$; but in each case it is a statement of light velocity having the magnitude of c regardless of its propagation direction:

$$dx^2 + dy^2 + dz^2 - c^2dt^2 = 0 \tag{E.28}$$

or equivalently

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = c^2. \tag{E.29}$$

(2.8) s is absolute because c is absolute: The previous exercise showed that c is absolute because s is absolute. Here you are asked to prove the converse statement: the constancy of c leads to the invariance of s. Of course in (2.30) we already demonstrated this invariance by a direct application of the Lorentz transformation, which is based on c's constancy. You are now asked to demonstrate this directly without any detailed Lorentz transformation calculations. Hint: From the vanishing invariant interval for light, $ds'^2 = ds^2 = 0$, and the fact that ds' and ds are infinitesimals of the same order, you can argue that the general intervals (not just for light) measured in two relative frames must be proportional to each other: $ds'^2 = Pds^2$, where P must be constant in space and time. From this, you can then show that the proportionality factor (which in principle be velocity-dependent) must be the identity, $P = 1$, by considering three frames $O \xrightarrow{\mathbf{v}} O' \xrightarrow{-\mathbf{v}} O''$, where the symbols above the arrows indicate the relative velocities.