The solutions manual
A College Course on Relativity and Cosmology by Ta-Pei Cheng
Chapter 4

**(4.1) Physical examples of $m_{\rm I}/m_{\rm G}$ dependence:** (a) For the frictionless inclined plane (with angle $\theta$) in Fig. 4.1a, find the acceleration's dependence on the ratio $m_{\rm I}/m_{\rm G}$. Thus a violation of the equivalence principle would show up as a material-dependence in the time required for a material block to slide down the plane. (b) For a simple pendulum (with string length $L$), Fig. 4.1b, find the oscillation period's dependence on the ratio $m_{\rm I}/m_{\rm G}$.

4.1A The dependence on the ratio $m_{\rm I}/m_{\rm G}$ in these two examples are shown below:

(a) Inclined plane: The $F = ma$ equation along the inclined plane, is $m_{\rm I}a = m_{\rm G}g\sin\theta$, leading to a material-dependent acceleration:

$$a_{\rm A,B} = g\sin\theta\left(\frac{m_{\rm G}}{m_{\rm I}}\right)_{\rm A,B}. \tag{E.98}$$

(b) Pendulum: For the simple pendulum with a light string of length $L$, we have

$$m_{\rm I}L\frac{d^2\theta}{dt^2} = -m_{\rm G}g\sin\theta. \tag{E.99}$$

This has the form of a simple harmonic oscillator equation (with $gm_{\rm G}/Lm_{\rm I}$ the angular frequency) when approximated by $\sin\theta\approx\theta$, leading to a period of

$$T_{\rm A} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}\left(\frac{m_{\rm I}}{m_{\rm G}}\right)_{\rm A}} \tag{E.100}$$

for a blob made of material A.

**(4.2) Two EP mind-teasers:** Even in mechanics in some instances the (weak) EP can be very useful in helping us to obtain physics results with simple analysis. Here are two notable examples: (a) Use the EP to explain the observation (see Fig. 4.2a) that a helium balloon *leans forward* in a (forward-) accelerating vehicle. (b) On his 76th birthday Einstein received a gift from his Princeton neighbor Eric Rogers. It was a toy composed of a ball attached by a spring to the inside of a bowl (a toilet plunger), which was just the right size to hold the ball. The upright bowl was fastened to a broomstick as in Fig 4.2b. What is the surefire way, suggested by the EP, to pop the ball back into the bowl each time?

**4.2A** The weak EP can be used to solve these two instances as follows:

(a) Forward leaning balloon: According to EP the effective gravity is the vector sum $\mathbf{g}_{\text{eff}} = \mathbf{g} + (-\mathbf{a})$, where $\mathbf{g}$ is the normal gravity (pointing vertically downward) while $\mathbf{a}$ is the acceleration of the vehicle. The buoyant force is always opposite to $\mathbf{g}_{\text{eff}}$.

(b) A toy for Einstein: The net force pulling the ball back into the bowl is the combination of gravity and spring restoring force. But the task can be made easy by dropping the whole contraption — EP informs us that gravity would disappear in this freely falling system. Without the interference of gravity, the spring will pull back the ball each time without any difficulty.

**(4.3) A GPS calculation:** We expect that the gravitational time dilation could be reduced if the satellite traveled at a lower orbit. Figure out how low the satellite's orbital radius ($r_{\text{S}}$) must be so that the time dilation effects due to gravity and due to relative motion *cancel* each other? What will be its period? Just as we did above, you may neglect $v_{\oplus}$.

(a) From the requirement that the total fractional time dilation $(\mathcal{F}_{\mathrm{mo}} + \mathcal{F}_{\mathrm{grav}}) = 0$ you should be able to deduce the relation of velocity $v_S$ to orbital radius $r_{\mathrm{S}}$:

$$v_S^2 = 2gr_\oplus \left(1 - \frac{r_\oplus}{r_S}\right). \tag{E.101}$$

Note that for this problem, it is entirely adequate to approximate the gamma factor as $\gamma \equiv \left(1-\beta^2\right)^{-1/2} \simeq 1 + \frac{1}{2}\beta^2$.

(b) From the centripetal acceleration equation, together with (E.101), one can then find the solution:

$$v_S^2 = \frac{2}{3}gr_\oplus \qquad \text{and} \qquad r_S = \frac{3}{2}r_\oplus. \tag{E.102}$$

(c) What will be resultant orbit period (in contrast to the real GPS system's 12 hour period)? From these numbers you should be able to conclude that such a system of low flying satellites may have some practical problems.

4.3A (a) The cancellation requirement leads to

$$-\gamma_S + 1 = \frac{\Phi_\oplus - \Phi_S}{c^2} \tag{E.103}$$

which becomes

$$-\left(1 + \frac{v_S^2}{2c^2}\right) + 1 = \frac{\Phi_\oplus}{c^2}\left(1 - \frac{1/r_S}{1/r_\oplus}\right). \tag{E.104}$$

This is just the displayed relation (E.101) given that $\Phi_\oplus = -gr_\oplus$.

(b) The centripetal acceleration equation being

$$\frac{GM}{r_S^2} = \frac{v_S^2}{r_S} \qquad \text{i.e.,} \qquad \frac{1}{r_S} = \frac{v_S^2}{gr_\oplus^2} \tag{E.105}$$

so that the relation (E.101) can now be written as

$$v_S^2 = 2gr_\oplus\left(1 - \frac{v_S^2}{gr_\oplus}\right) \qquad \text{i.e.,} \qquad v_S^2 = \frac{2}{3}gr_\oplus. \tag{E.106}$$

Plug this back into the previous equation (E.105), we immediately obtain $r_S = 3r_\oplus/2$.

(c) Knowing the results of $v_S$ and $r_S$, the period can then be calculated

$$P = \frac{2\pi r_S}{v_S} = 3\pi\sqrt{\frac{3}{2}\frac{r_\oplus}{g}} \simeq 3\ hr. \tag{E.107}$$