The solutions manual - Chapters 5 & 6
A College Course on Relativity and Cosmology by Ta-Pei Cheng

(5.1) Coordinate choice: Clearly the ideal choice of coordinate system often depends on the task at hand. Consider the calculation in the space of a 2D plane of the circumference of a circle of radius R ($2\pi R$ of course). It is easy in polar coordinates (r, θ), but rather complicated in Cartesian coordinates (x, y). Carry out the calculations in both coordinate systems.

5.1A (a) In polar coordinates (r, θ), we have for $r = R$

$$s = \int ds = \int_0^{2\pi} R d\theta = 2\pi R. \tag{E.108}$$

(b) In Cartesian system (x, y) with origin at the center of the circle,

$$x^2 + y^2 = R^2, \tag{E.109}$$

we just calculate the circumference of the first quadrant with both coordinates having the range of $(0, R)$:

$$s_4 = \int \sqrt{dx^2 + dy^2} = \int dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{E.110}$$

Differentiating (E.109) we have

$$\frac{dy}{dx} = -\frac{x}{y} = -\frac{x}{\sqrt{R^2 - x^2}} \tag{E.111}$$

leading to

$$s = 4s_4 = 4R \int_0^R \frac{dx}{\sqrt{R^2 - x^2}} = 4R \left[\sin^{-1} \frac{x}{R}\right]_0^R = 2\pi R. \tag{E.112}$$

(5.2) Cylindrical coordinate metric: Find the metric tensor for the cylindrical coordinates (ρ, ϕ) on a 2-sphere. Suggestion: From Fig. 5.1 note that the radial coordinate is related to the polar angle by $\rho = R \sin\theta$; then show that

$$g_{ab}^{(\rho,\phi)} = \begin{pmatrix} R^2/\left(R^2 - \rho^2\right) & 0 \\ 0 & \rho^2 \end{pmatrix}. \tag{E.113}$$

5.2A Since the cylindrical radial coordinate is related to the polar angle by $\rho = R\sin\theta$, we have $d\rho = R\cos\theta d\theta = \sqrt{1 - \sin^2\theta} R d\theta$, thus $d\rho^2 = \left(1 - \rho^2/R^2\right) R^2 d\theta^2$. Sub this into the polar coordinate metric

$$\begin{aligned} [ds^2]_{(\theta,\phi)} &= R^2 d\theta^2 + R^2 \sin^2\theta d\phi^2 \\ &= \frac{R^2}{R^2 - \rho^2} d\rho^2 + \rho^2 d\phi^2 = [ds^2]_{(\rho,\phi)}. \end{aligned} \tag{E.114}$$

(5.3) Transformation in curved space: Find the coordinate transformation matrix $[\Lambda]$ (i.e., showing its coordinate-dependence) that changes the polar coordinates (θ, ϕ) to the cylindrical (ρ, ϕ).

$$\begin{pmatrix} d\rho \\ d\phi \end{pmatrix} = [\Lambda] \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}. \tag{E.115}$$

5.3A Since one of the coordinates is unchanged, clearly we have $[\Lambda]_{\phi\phi} = 1$ and $[\Lambda]_{\rho\phi} = [\Lambda]_{\phi\theta} = 0$. From Ex.(5.2) we see that $d\rho = R\cos\theta d\theta$, hence $[\Lambda]_{\rho\theta} = R\cos\theta$. In this way we have the transformation

$$\begin{pmatrix} d\rho \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}. \tag{E.116}$$

(5.4) Geodesics on simple surfaces: Use the geodesic equation.(5.25) to confirm the familiar results that the geodesic is (a) a straight line on a flat plane and (b) a great circle on a spherical surface. Suggestion: For the case (b), working out the full parametrization can be complicated; just check that the great circle given by $\phi =$ constant and $\theta = \alpha + \beta\tau$ solves the relevant geodesic equation.

5.4A (a) Flat plane: For this 2D space with Cartesian coordinates $(x^1, x^2) = (x, y)$, the metric $g_{ab} = \delta_{ab}$. The second term in the geodesic equation (5.25) vanishes, as well as the two components of the equation $d\dot{x}^\nu/d\sigma$

$$\ddot{x} = 0 \quad \text{and} \quad \ddot{y} = 0, \tag{E.117}$$

which have respective solutions of

$$x = \mathrm{A} + \mathrm{B}\sigma \quad \text{and} \quad y = \mathrm{C} + \mathrm{D}\sigma. \tag{E.118}$$

They can be combined as

$$y = \alpha + \beta x \tag{E.119}$$

with (A, B, C, D) and (α, β) being constants. We recognize this as the equation for a straight line.

(b) Spherical surface: For a 2-sphere, we choose the coordinates $(x^1, x^2) = (\theta, \phi)$ with a metric given by (5.8) For the θ component of the geodesic equation (5.25) is

$$\ddot{\theta} = \sin\theta\cos\theta\dot{\phi}^2, \tag{E.120}$$

the ϕ component equation,

$$2\sin\theta\cos\theta\dot{\theta}\dot{\phi} + \sin^2\theta\ddot{\phi} = 0. \tag{E.121}$$

Instead of working out the full parametric solution, we will just check that $\phi =$ constant and $\theta = \alpha + \beta\sigma$ solve these two equations. Clearly these solutions describe longitudinal great circles on the sphere

(6.1) Gaussian curvature is coordinate-independent: Check the coordinate-independence of the curvature (a) for a flat plan in Cartesian and in polar coordinates, and (b) for a spherical surface with radius R in polar coordinates and in cylindrical coordinates, by plugging in their respective metrics, (6.6), (5.8) and (5.9), into the curvature formula (6.7).

6.1A (a) For Cartesian coordinates, the metric is constant; so clearly we have $K_{k=0}^{(x,y)} = 0$. If we use a polar coordinates (r, θ) with $g_{11} = 1$ and $g_{22} = r^2$ the curvature formula (6.7) yields

$$K_{k=0}^{(r,\theta)} = \frac{1}{2r^2}\left\{-2 + \frac{1}{2r^2}\left[0 + 4r^2\right]\right\} = 0. \tag{E.122}$$

(b) For sphere with the polar coordinates (θ, ϕ) of a spherical surface, we have from (5.8) $g_{11} = R^2$ and $g_{22} = R^2 \sin^2\theta$, the formula (6.7) yields

$$\begin{aligned} K_{k=1}^{(\theta,\phi)} &= \frac{1}{2R^4\sin^2\theta}\left\{0 - 2R^2\cos 2\theta + 0 + \frac{1}{2R^2\sin^2\theta}\left[0 + R^4\sin^2 2\theta\right]\right\} \\ &= \frac{1}{2R^2\sin^2\theta}\left\{-2R^2\left(\cos^2\theta - \sin^2\theta\right) + \frac{1}{2R^2}R^4 4\cos^2\theta\right\} = \frac{1}{R^2}; \end{aligned} \tag{E.123}$$

For the cylinder coordinates (ρ, ϕ) on a spherical surface, we have from and (5.9). $g_{11} = R^2/(R^2 - \rho^2)$ and $g_{22} = \rho^2$. This leads to the same result:

$$K_{k=1}^{(\rho,\phi)} = \frac{(R^2 - \rho^2)}{2R^2\rho^2}\left\{0 - 2 + \left[\frac{2\rho^2}{(R^2 - \rho^2)} + 0\right] + \frac{1}{2\rho^2}\left[0 + 4\rho^2\right]\right\} = \frac{1}{R^2}. \tag{E.124}$$

(6.2) Pseudosphere metric from embedding coordinates: (i) Recover the result (5.8) for the metric g_{ab} of a 2D $k = +1$ surface in the Gaussian coordinates $\left(x^{1,2} = \theta, \phi\right)$ by way of its 3D embedding coordinates $X^i(\theta, \phi)$ as shown in (5.3). From the invariant interval in the 3D embedding space, extended to the Gaussian coordinate differentials by the chain rule of differentiation, we have

$$ds^2 = \delta_{ij} dX^i dX^j = \delta_{ij}\frac{\partial X^i}{\partial x^a}\frac{\partial X^j}{\partial x^b}dx^a dx^b. \tag{E.125}$$

By comparing this to (5.5) you can identify the metric for the 2D space:

$$g_{ab} = \delta_{ij}\frac{\partial X^i}{\partial x^a}\frac{\partial X^j}{\partial x^b}, \tag{E.126}$$

which can be viewed as the transformation of the metric from Cartesian to the polar coordinates (ii) For a $k = -1$ 2D pseudosphere its hyperbolic Gaussian coordinates $\left(x^{1,2} = \psi, \phi\right)$ can be related to the 3D embedding coordinates, in analogy with (5.3), by

$$X^1 = R\sinh\psi\cos\phi, \quad X^2 = R\sinh\psi\sin\phi, \quad X^3 = R\cosh\psi. \tag{E.127}$$

This embedding space has the metric $\eta_{ij} = \text{diag}(1, 1, -1)$. Follow the above steps to deduce the 2D space metric of

$$g_{ab}^{(\psi,\phi)} = R^2 \begin{pmatrix} 1 & 0 \\ 0 & \sinh^2 \end{pmatrix}. \tag{E.128}$$

(iii) Show that this metric leads, via (6.7), to a negative Gaussian curvature of $K = -1/R^2$. (iv) Show that while the circumference of a circle with radius r on a spherical surface is $2\pi R \sin(r/R)$, which is smaller than $2\pi r$ as in Fig. 6.1(a), on a hypersphere it is $2\pi R \sinh(r/R)$, and hence greater than $2\pi r$. (v) A $k = +1$ 2D sphere has no boundary but a finite area: $A_2 = \int ds_\theta ds_\phi = 4\pi R^2$. Similarly demonstrate that a $k = -1$ 2D pseudosphere has no boundary but an area that is infinite: $\tilde{A}_2 = \int ds_\psi ds_\phi = \infty$.

6.2A i. For 2-sphere in polar coordinates (θ, ϕ) :

$$\begin{aligned} g_{11} &= \frac{\partial X^1}{\partial x^1}\frac{\partial X^1}{\partial x^1} + \frac{\partial X^2}{\partial x^1}\frac{\partial X^2}{\partial x^1} + \frac{\partial X^3}{\partial x^1}\frac{\partial X^3}{\partial x^1} \\ &= R^2\cos^2\theta\cos^2\phi + R^2\cos^2\theta\sin^2\phi + R^2\sin^2\theta = R^2. \end{aligned} \tag{E.129}$$

and

$$\begin{aligned} g_{22} &= \frac{\partial X^1}{\partial x^2}\frac{\partial X^1}{\partial x^2} + \frac{\partial X^2}{\partial x^2}\frac{\partial X^2}{\partial x2} + \frac{\partial X^3}{\partial x^2}\frac{\partial X^3}{\partial x^2} \\ &= R^2\sin^2\theta\sin^2\phi + R^2\sin^2\theta\cos^2\phi = R^2\sin^2\theta. \end{aligned} \tag{E.130}$$

ii. For 2-pseudosphere in hyperbolic coordinates (ψ, ϕ) as shown in (E.127):

$$\begin{aligned} g_{11} &= \frac{\partial X^1}{\partial x^1}\frac{\partial X^1}{\partial x^1} + \frac{\partial X^2}{\partial x^1}\frac{\partial X^2}{\partial x^1} - \frac{\partial X^3}{\partial x^1}\frac{\partial X^3}{\partial x^1} \\ &= R^2\cosh^2\theta\cos^2\phi + R^2\cosh^2\theta\sin^2\phi - R^2\sinh^2\theta = R^2. \end{aligned} \tag{E.131}$$

and

$$\begin{aligned} g_{22} &= \frac{\partial X^1}{\partial x^2}\frac{\partial X^1}{\partial x^2} + \frac{\partial X^2}{\partial x^2}\frac{\partial X^2}{\partial x2} - \frac{\partial X^3}{\partial x^2}\frac{\partial X^3}{\partial x^2} \\ &= R^2\sinh^2\theta\sin^2\phi + R^2\sinh^2\theta\cos^2\phi = R^2\sinh^2\theta. \end{aligned} \tag{E.132}$$

iii. For the hyperbolic coordinates (ψ, ϕ) on a 2D surface, we have, from $g_{11} = R^2$ and $g_{22} = R^2\sinh^2\theta$, the Gaussian curvature (6.7):

$$\begin{aligned} K &= \frac{1}{2R^4\sinh^2\theta}\left\{0 - 2R^2\cosh 2\theta + 0 + \frac{1}{2R^2\sinh^2\theta}\left[0 + R^4\sinh^2 2\theta\right]\right\} \\ &= \frac{1}{2R^2\sinh^2\theta}\left\{-2R^2\left(\cosh^2\theta + \sinh^2\theta\right) + \frac{1}{2R^2}R^4\cosh^{2=\pi/s}\theta\right\} = -\frac{1}{R^2}. \end{aligned} \tag{E.133}$$

iv. Given (E.129) we have

$$ds_\theta^2 = g_{11}d\theta^2 = R^2 d\theta^2, \qquad ds_\phi^2 = g_{22}d\phi^2 = R^2\sin^2\theta d\phi^2 \tag{E.134}$$

we have the circumference of a circle on a 2-sphere

$$C_2 = \int ds_\phi = R\sin\theta\int_0^{2\pi} d\phi = 2\pi R\sin\left(\frac{r}{R}\right); \tag{E.135}$$

on a 2-pseudosphere

$$\tilde{C}_2 = \int ds_\phi = R\sinh\theta\int_0^{2\pi} d\phi = 2\pi R\sinh\left(\frac{r}{R}\right) \tag{E.136}$$

v. The area of a 2D sphere with radius R

$$A_2 = \int ds_\theta ds_\phi = R^2\int_{-1}^{1} d\cos\theta\int_0^{2\pi} d\phi = 4\pi R^2. \tag{E.137}$$

Similarly for the 2D hypersphere with

$$ds_\psi^2 = R^2 d\psi^2, \qquad ds_\phi^2 = R^2\sinh^2\psi d\phi^2 \tag{E.138}$$

we have

$$\tilde{A}_2 = \int ds_\psi ds_\phi = R^2\int_1^{\infty} d\cosh\psi\int_0^{2\pi} d\phi = \infty. \tag{E.139}$$

(6.3) Light deflection from solving the geodesic equation: Take the following steps to obtain the bending of light result shown in (6.77): (a) Identify the two constants of motion. (b) Express the $L = 0$ equation in terms of these constants. (c) Changing the curve parameter differential $d\tau \to d\phi$ and by changing the radial distance variable to its inverse $u \equiv 1/r$, you should find the light trajectory to obey

$$u'' + u - \epsilon u^2 = 0, \tag{E.140}$$

where $u'' = d^2u/d\phi^2$ and $\epsilon = 3r^*/2$. (d) Solve (E.140) by perturbation: $u = u_0 + \epsilon u_1$. Suggestion: Parameterize the first-order perturbation solution as $u_1 = \alpha + \beta\cos 2\phi$; then fix the constants α and β. (e) From this solution of the orbit $r(\phi)$ for the light trajectory, one can deduce the angular deflection $\delta\phi$ result of (6.77) by comparing the directions of the initial and final asymptotes $\left(r = \infty \text{ at } \phi_i = \pi/2 + \delta\phi/2, \text{ and } \phi_f = -\pi/2 - \delta\phi/2\right)$ as shown in Fig. 6.12(b).

6.3A To obtain the result (6.77) via the geodesic equation, we take the following steps.

(a) For a fixed $x^\alpha = (ct, \phi)$,We have two constants of motion $\partial L/\partial \dot{x}^\alpha$ with $L = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$. For $x^\alpha = x^0$, we have $g_{\mu\nu}\partial\left(\dot{x}^\mu \dot{x}^\nu\right)/\partial \dot{x}^0 = 2g_{00}\dot{x}^0$; similarly for $x^\alpha = \phi$, we have $g_{\phi\phi}\dot{\phi}$. Since $g_{00} = -\left(1 - r^*/r\right)$ and $g_{\phi\phi} = r^2 \sin^2\theta = r^2$ as θ is fixed to be $\pi/2$, we call these two constants $\kappa \equiv \left(1 - r^*/r\right) c\dot{t} = -g_{00}\dot{x}^0$; and $j \equiv r^2\dot{\phi}$.

(b) Express the RHS of (6.79) in terms of κ and j in the $L = 0$ equation: $L = g_{00}\left(\dot{x}^0\right)^2 - \dot{r}^2/g_{00} + g_{\phi\phi}\dot{\phi}^2 = \kappa^2/g_{00} - \dot{r}^2/g_{00} + r^2\dot{\phi}^2 = 0$ which can be written as

$$\dot{r}^2 + \frac{j^2}{r^2}\left(1 - \frac{r^*}{r}\right) = \kappa^2. \tag{E.141}$$

(c) Since $jd\tau = r^2 d\phi$ or $1/d\tau = j/\left(r^2 d\phi\right)$, we get

$$\dot{r} = \frac{dr}{d\tau} = \frac{j}{r^2}\frac{dr}{d\phi}. \tag{E.142}$$

Then with $u = 1/r$,

$$u' = \frac{dr^{-1}}{d\phi} = -\frac{1}{r^2}\frac{dr}{d\phi} = -\frac{\dot{r}}{j}. \tag{E.143}$$

or $\left(u'\right)^2 = \dot{r}^2/j^2$ so that (E.141) may be re-written as

$$\left(u'\right)^2 + u^2 - r^* u^3 = \frac{\kappa^2}{j^2}. \tag{E.144}$$

By a simple differentiation with respect to ϕ, this equation becomes

$$u'' + u - \epsilon u^2 = 0, \qquad \text{with } \epsilon = 3r^*/2. \tag{E.145}$$

(d) Solve (E.145) by perturbation $u = u_0 + \epsilon u_1$.

$$\left(u_0'' + u_0\right) + \epsilon\left(u_1'' + u_1 - u_0^2\right) = 0. \tag{E.146}$$

The zeroth order equation $u_0'' = -u_0$ is a simple harmonic oscillator equation with unit angular frequency and has the solution

$$u_0 = \frac{\cos\phi}{r_{\min}}, \tag{E.147}$$

which is a straight-line $r_0 = r_{\min}/\cos\phi$ going from $r_0 = \infty$ with $\phi_i = \pi/2$ to $r_0 = \infty$ with $\phi_f = -\pi/2$ as shown in Fig. 6.12(a). For the first order equation

$$\frac{d^2 u_1}{d\phi^2} + u_1 - \frac{1 + \cos 2\phi}{2r_{\min}^2} = 0 \tag{E.148}$$

we try the form of $u_1 = \alpha + \beta\cos 2\phi$ so that

$$-4\beta\cos 2\phi + \alpha + \beta\cos 2\phi - \frac{1}{2r_{\min}^2} - \frac{\cos 2\phi}{2r_{\min}^2} = 0, \tag{E.149}$$

which fixes the constants $\alpha = 1/2r_{\min}^2$, $\beta = -1/6r_{\min}^2$. In this way one finds the result, accurate up to the first order in r^*, of a bent trajectory:

$$\frac{1}{r} = \frac{\cos\phi}{r_{\min}} + \frac{r^*}{r_{\min}^2}\frac{3-\cos 2\phi}{4}. \tag{E.150}$$

(e) From this expression for the light trajectory $r(\phi)$, one can deduce the angular deflection $\delta\phi$, cf. Fig. 6.12(b), by plugging in (either the initial or final) asymptote $r = \infty$, and $\phi_i = \pi/2 + \delta\phi/2$

$$0 \simeq -\frac{\sin\delta\phi/2}{r_{\min}} + \frac{r^*}{r_{\min}^2}\frac{3+1}{4} \simeq \frac{-1}{r_{\min}}\left(\frac{\delta\phi}{2} - \frac{r^*}{r_{\min}}\right), \tag{E.151}$$

which leads to the result of (6.77) of $\delta\phi = 2r^*/r_{\min}$.

7.2A Here one wants the outgoing light geodesics be represented by 45^o worldlines $cd\tilde{t} = dr$, instead of $cd\bar{t} = -dr$.This suggests that in the $ds^2 = 0$