Chapter 11 Solutions
A College Course on Relativity and Cosmology by Ta-Pei Cheng
Appendix E

INCORRECT FONT FOR CHRISTOFFL SYMBOLS ON THIS PAGE !!

(11.1) Christoffel symbols as the metric tensor derivative: (a) The geometry in which we are working has the property that two covariant di ffeven-tiation operations on a scalar tensor commute: $D_\mu D_\nu$S= $D_\nu D_\mu$S. (We call such derivatives torsion-free.) From this prove that Christoffel sym-bols are symmetric with respect to the interchange of their lower indices: $_{\nu\lambda} = \Omega^{\mu}_{\lambda\nu}$

(b) From the relation (11.22) of Christoffel symbols as the

coe cients of expansion of the derivative $\partial_\nu \mathbf{e}_\mu$, we showed that the metric is covariantly constant as in (11.29). From this, derive the expression for Christoffe symbols,as the first derivatives of the metric tensor, shown in (5.30). To signify its importance, this relation is called the *fundamental theorem of Riemannian geometry.* Suggestion: One can obtain the result by taking the linear combination of three equations expressing ($Dg = 0$) with indices cyclically permuted and by using $\Omega^{\mu}_{\nu\lambda} = \Omega^{\mu}_{\lambda\nu}$ as shown in part (a).

11.1A (a) From $D_\mu D_\nu \Omega = D_\nu D_\mu \Omega$,we immediately have $D_\mu (\partial_\nu \Omega) = D_\nu (\partial_\mu \Omega)$ because Ω is a scalar. On the other hand, the derivatives $(\partial_\nu \Omega)$ and $(\partial_\mu \Omega)$ are rank 1 covariant vectors so their covariant derivatives involve the Christoffel symbols:

$$\partial_\mu \partial_\nu \Omega - \Omega^{\lambda}_{\mu\nu} \partial_\lambda \Omega = \partial_\nu \partial_\mu \Omega - \Omega^{\lambda}_{\nu\mu} \partial_\lambda \Omega.$$

From the commutativity of the ordinary differentiation we get the claimed result,

$$\Omega^{\lambda}_{\mu\nu} = \Omega^{\lambda}_{\nu\mu}.$$

(b) We start by using several versions of (11.29) with their indices permuted cyclically:

$$D_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Omega^{\rho}_{\lambda\mu} g_{\rho\nu} - \Omega^{\rho}_{\lambda\nu} g_{\mu\rho} = 0$$
$$D_\nu g_{\lambda\mu} = \partial_\nu g_{\lambda\mu} - \Omega^{\rho}_{\nu\lambda} g_{\rho\mu} - \Omega^{\rho}_{\nu\mu} g_{\lambda\rho} = 0 \quad \text{(E.186)}$$
$$-D_\mu g_{\nu\lambda} = -\partial_\mu g_{\nu\lambda} + \Omega^{\rho}_{\mu\nu} g_{\rho\lambda} + \Omega^{\rho}_{\mu\lambda} g_{\nu\rho} = 0$$

Adding up these three equations and using the symmetry property of $\Omega^{\rho}_{\mu\nu} = \Omega^{\rho}_{\nu\mu}$ derived in (a), we obtain:

$$\partial_\lambda g_{\mu\nu} + \partial_\nu g_{\lambda\mu} - \partial_\mu g_{\nu\lambda} - 2\Omega^{\rho}_{\lambda\nu} g_{\mu\rho} = 0 \text{ or,} \tag{E.187}$$

in its equivalent form,

$$\Omega^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left[\partial_\nu g_{\mu\rho} + \partial_\mu g_{\nu\rho} - \partial_\rho g_{\mu\nu}\right], \tag{E.188}$$

as

rst displayed in (5.30).

(11.2) Parallel-transported of a vector along a geodesic: Show that when a vector A_μ is parallel-transported along a geodesic, the angle between the vector and the geodesic (*i.e.*, the tangent of the geodesic) as in (11.37) is unchanged as in Fig. 11.2(b). Namely, prove the following relation:

$$\frac{D}{d\sigma}\left(A_\mu \frac{dx^\mu}{d\sigma}\right) = 0. \tag{E.189}$$

11.2A The proof is straightforward:

$$\frac{D}{d\sigma}\left(A_\mu \frac{dx^\mu}{d\sigma}\right) = \frac{DA_\mu}{d\sigma}\left(\frac{dx^\mu}{d\sigma}\right) + A_\mu \frac{D}{d\sigma}\left(\frac{dx^\mu}{d\sigma}\right). \tag{E.190}$$

The RHS indeed vanishes: the first term is zero because we are parallel transport the vector, cf. (11.6),

$$\frac{DA^\mu}{d\sigma} = 0; \tag{E.191}$$

the second term is zero because the curve is a geodesic satisfying (11.37),

$$\frac{D}{d\sigma}\left(\frac{dx^\mu}{d\sigma}\right) = 0. \tag{E.192}$$

(11.3) Riemann curvature tensor as the commutator of covariant derivatives: We can obtain the same result as in Box 11.1 somewhat more efficiently by calculating the double covariant derivative

$$D_\alpha D_\beta A^\mu = D_\alpha \left(\partial_\beta A^\mu + \Gamma^\mu_{\beta\lambda} A^\lambda\right) = ..., \tag{E.193}$$

as well as the reverse order $D_\beta D_\alpha A^\mu = D_\beta \left(\partial_\alpha A^\mu + \Gamma^\mu_{\alpha\lambda} A^\lambda\right) = ...$ Show that their difference (expressed here as a commutator) is just the expression for the Riemann tensor given by (11.40):

$$\left[D_\alpha, D_\beta\right] A^\mu = R^\mu{}_{\lambda\alpha\beta} A^\lambda. \tag{E.194}$$

11.3A Following the rule of (11.25), we have

$$\begin{aligned} D_\alpha D_\beta A_\mu &= \partial_\alpha \left(D_\beta A_\mu\right) - \underbrace{\Gamma^\nu_{\alpha\beta} D_\nu A_\mu}_{\text{drop}} - \Gamma^\nu_{\alpha\mu} D_\beta A_\nu \\ &= \underbrace{\partial_\alpha \partial_\beta A_\mu}_{\text{drop}} - \partial_\alpha \left(\Gamma^\nu_{\beta\mu} A_\nu\right) - \Gamma^\nu_{\alpha\mu} \partial_\beta A_\nu + \Gamma^\nu_{\alpha\mu} \Gamma^\lambda_{\beta\nu} A_\lambda \\ &= -\left(\partial_\alpha \Gamma^\lambda_{\beta\mu}\right) A_\lambda \underbrace{- \Gamma^\nu_{\beta\mu} \partial_\alpha A_\nu - \Gamma^\nu_{\alpha\mu} \partial_\beta A_\nu}_{\text{drop}} + \Gamma^\nu_{\alpha\mu} \Gamma^\lambda_{\beta\nu} A_\lambda \end{aligned} \tag{E.195}$$

The underlined terms are symmetric in the indices (α, β) and will be cancelled when we include the $-D_\beta D_\alpha A_\mu$ calculation. From this we clearly get (11.56) with $R^\lambda{}_{\mu\alpha\beta}$ given by (11.40).

(11.4) From geodesic deviation to nonrelativistic tidal forces: Show that the equation of geodesic deviation (11.67) reduces to Newtonian deviation equation (6.24) in the Newtonian limit. In the nonrelativistic limit of slow moving particle with 4-velocity of $dx^\alpha/d\tau \simeq (c,0,0,0)$, the GR equation (11.67) is reduced to

$$\frac{d^2s^i}{dt^2} = -c^2 R^i{}_{0j0} s^j. \tag{E.196}$$

We have also set $s^0 = 0$ because we are comparing the two particle's acceleration at the same time. Thus (6.24) can be recovered by showing the relation

$$R^i{}_{0j0} = \frac{1}{c^2}\frac{\partial^2\Phi}{\partial x^i \partial x^j} \tag{E.197}$$

in the Newtonian limit. You are asked to prove the this limit expression of the Riemann curvature (11.40).

11.4A Besides slow moving particles, the Newtonian limit means a weak gravitational field: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $h_{\mu\nu}$ being small. Thus (5.30) becomes

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}\eta^{\mu\rho}[\partial_\alpha h_{\beta\rho} + \partial_\beta h_{\alpha\rho} - \partial_\rho h_{\alpha\beta}]. \tag{E.198}$$

Also, in this weak-field limit, we can drop the quadratic terms ($\Gamma\Gamma$) in the curvature (11.40) so that there are only two terms, related by the interchange of (β, λ) indices

$$\begin{aligned} R^\mu_{\alpha\lambda\beta} &= \partial_\lambda\Gamma^\mu_{\alpha\beta} - \partial_\beta\Gamma^\mu_{\lambda\alpha} \\ &= \frac{1}{2}\eta^{\mu\rho}[\partial_\lambda\partial_\alpha h_{\beta\rho} - \partial_\lambda\partial_\rho h_{\alpha\beta} - \partial_\beta\partial_\alpha h_{\lambda\rho} + \partial_\beta\partial_\rho h_{\alpha\lambda}] \end{aligned} \tag{E.199}$$

after cancelling two terms. Thus

$$R^i_{0j0} = \frac{1}{2}[\partial_j\partial_0 h_{0i} - \partial_j\partial_i h_{00} - \partial_0\partial_0 h_{ji} + \partial_0\partial_i h_{0j}] = -\frac{1}{2}\partial_i\partial_j h_{00}. \tag{E.200}$$

Because the Newtonian limit also has the static field condition, to reach the last line we have dropped all time derivatives ∂_0's. With $h_{00} = -2\Phi/c^2$ as given by (5.41), we have the sought-after relation of

$$R^i_{0j0} = \frac{1}{c^2}\frac{\partial^2\Phi}{\partial x^i \partial x^j}. \tag{E.201}$$

(11.5) Jacobi identities and double commutator of covariant derivatives: (a) Prove the Jacobi identity. Namely, demonstrate explicitly that the cyclic combination of three double commutators in (11.72) vanishes. (b) Use the expression of Riemann tensor in terms of the double commutator as in (11.57) to show

$$[D_\lambda, [D_\mu, D_\nu]]\, A_\alpha = -\left(D_\lambda R^\gamma{}_{\alpha\mu\nu}\right) A_\gamma + R^\gamma{}_{\lambda\mu\nu} D_\gamma A_\alpha. \tag{E.202}$$

11.5A (a) By expanding out the double commutators, the operator relation of Jacobi identity is seen to be valid:

$$\begin{aligned} & [D_\lambda,[D_\mu,D_\nu]]+[D_\nu,[D_\lambda,D_\mu]]+[D_\mu,[D_\nu,D_\lambda]] \\ = & D_\lambda D_\mu D_\nu - D_\lambda D_\nu D_\mu - D_\mu D_\nu D_\lambda + D_\nu D_\mu D_\lambda + \\ & D_\nu D_\lambda D_\mu - D_\nu D_\mu D_\lambda - D_\lambda D_\mu D_\nu + D_\mu D_\lambda D_\nu + \\ & D_\mu D_\nu D_\lambda - D_\mu D_\lambda D_\nu - D_\nu D_\lambda D_\mu + D_\lambda D_\nu D_\mu \\ = & 0 \end{aligned} \tag{E.203}$$

(b) Using the expression of the Riemann tensor in terms of commutator of covariant derivatives on a covariant vector

$$[D_\alpha,D_\beta]\,A_\mu=-R^\lambda{}_{\mu\alpha\beta}A_\lambda \tag{E.204}$$

and on a rank-2 tensor

$$[D_\alpha,D_\beta]\,B_{\lambda\nu}=-R^\gamma{}_{\nu\alpha\beta}B_{\lambda\gamma}-R^\gamma{}_{\lambda\alpha\beta}B_{\gamma\nu}, \tag{E.205}$$

we have

$$\begin{aligned} [D_\lambda,[D_\mu,D_\nu]]\,A_\alpha &= D_\lambda\,[D_\mu,D_\nu]\,A_\alpha-[D_\mu,D_\nu]\,D_\lambda A_\alpha \\ &= -D_\lambda\left(R^\gamma{}_{\alpha\mu\nu}A_\gamma\right)+R^\gamma{}_{\alpha\mu\nu}D_\lambda A_\gamma+R^\gamma{}_{\lambda\mu\nu}D_\gamma A_\alpha \\ &= -D_\lambda R^\gamma{}_{\alpha\mu\nu}A_\gamma-R^\gamma{}_{\alpha\mu\nu}D_\lambda A_\gamma+R^\gamma{}_{\alpha\mu\nu}D_\lambda A_\gamma+R^\gamma{}_{\lambda\mu\nu}D_\gamma A_\alpha \\ &= -D_\lambda R^\gamma{}_{\alpha\mu\nu}A_\gamma+R^\gamma{}_{\lambda\mu\nu}D_\gamma A_\alpha, \end{aligned} \tag{E.206}$$

where (11.57) has been applied to rank-2 tensor $D_\lambda A_\alpha$, then the middle two terms cancel.

(11.6) Newtonian limit of Einstein field equation: Show that Newton's gravitational field law, written in differential form (4.7), is the leading-order approximation to the Einstein field equation (11.88) in the Newtonian limit (cf Section 5.3.1) for a slow ($v \ll c$) source particle producing a static and weak gravitational field. In this way, one can also establish the connection between the proportionality constant κ and Newton's constant as shown in (6.39).

11.6A

- Slow moving source particle In the non-relativistic regime of small v/c, the rest energy density term T_{00} being dominant, we shall concentrate on the 00-component of (11.88), as other terms are down by $O\,(v/c)$:

$$R_{00}=k\left(T_{00}-\frac{1}{2}Tg_{00}\right) \tag{E.207}$$

with

$$T=g^{\mu\nu}T_{\mu\nu}\backsimeq g^{00}T_{00}=\frac{1}{g_{00}}T_{00}. \tag{E.208}$$

(E.207) becomes

$$R_{00} = \frac{1}{2}\kappa T_{00}. \tag{E.209}$$

To recover the Newtonian field equation, we need to show that $R_{00} \to \bigtriangledown^2 g_{00}$: From the definition of Ricci tensor (in terms of the Riemann-Christoffel tensor), we have

$$R_{00} = g^{\mu\nu} R_{\mu 0 \nu 0} = g^{ij} R_{i0j0} \tag{E.210}$$

where $(i = 1, 2, 3)$ and in reaching the last equality we have used the fact that the tensor components such as R_{0000} and R_{i000} all vanish because of symmetry properties of the curvature tensor, $R_{\mu\nu\lambda\rho} = -R_{\mu\nu\rho\lambda}$, etc.

- Weak field limit The Newtonian limit also corresponds to weak field limit, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $h_{\mu\nu}$ being small. Since $\partial g = \partial h$, we will keep as few powers of ∂g as possible: i.e., keep $\partial\partial g$ terms rather than $(\partial g)^2$s, etc.

$$R_{\mu\nu\alpha\beta} = -\frac{1}{2}\left(\partial_\mu\partial_\alpha g_{\nu\beta} - \partial_\nu\partial_\alpha g_{\mu\beta} + \partial_\nu\partial_\beta g_{\mu\alpha} - \partial_\mu\partial_\beta g_{\nu\alpha}\right). \tag{E.211}$$

Substitute this into (E.210) we have

$$R_{00} = g^{ij} R_{i0j0} = -\frac{g^{ij}}{2}\left(\partial_i\partial_j g_{00} - \partial_0\partial_j g_{i0} + \partial_0\partial_0 g_{ij} - \partial_i\partial_0 g_{0j}\right) \tag{E.212}$$

- Static limit Newtonian limit also corresponds to a static situation, we can drop in (E.212) all terms having a time derivative ∂_0 factor,

$$R_{00} = -\frac{1}{2}\nabla^2 g_{00}. \tag{E.213}$$

After using the relation (5.41) between g_{00} and the Newtonian potential Φ and $T_{00} = \rho c^2$as discussed in Chapter 3 following (3.84), in this way (E.209) becomes

$$\frac{1}{2}\nabla^2\left(1 + 2\frac{\Phi}{c^2}\right) = \frac{1}{2}\kappa\rho c^2, \tag{E.214}$$

or

$$\nabla^2\Phi = \frac{1}{2}\kappa\rho c^4. \tag{E.215}$$

Thus we see that the Einstein equation indeed has the correct Newtonian limit of $\nabla^2\Phi = 4\pi G_\mathrm{N}\rho$ when we identify the proportional constant as in (6.39).