The solutions manual - Chapter 7
A College Course on Relativity and Cosmology by Ta-Pei Cheng

which fixes the constants $\alpha = 1/2r_{\rm min}^2$, $\beta = -1/6r_{\rm min}^2$. In this way one finds the result, accurate up to the first order in r^*, of a bent trajectory:

$$\frac{1}{r} = \frac{\cos\phi}{r_{\rm min}} + \frac{r^*}{r_{\rm min}^2}\frac{3-\cos 2\phi}{4}. \tag{E.150}$$

(e) From this expression for the light trajectory $r(\phi)$, one can deduce the angular deflection $\delta\phi$, cf. Fig. 6.12(b), by plugging in (either the initial or final) asymptote $r = \infty$, and $\phi_i = \pi/2 + \delta\phi/2$

$$0 \simeq -\frac{\sin\delta\phi/2}{r_{\rm min}} + \frac{r^*}{r_{\rm min}^2}\frac{3+1}{4} \simeq \frac{-1}{r_{\rm min}}\left(\frac{\delta\phi}{2} - \frac{r^*}{r_{\rm min}}\right), \tag{E.151}$$

which leads to the result of (6.77) of $\delta\phi = 2r^*/r_{\rm min}$.

(7.1) The travel time from horizon to the singular origin: (a) How much proper time $\Delta\tau$ (in terms of the Schwarzschild radius r^*) passes for a probe falling from the event horizon to the $r = 0$ singularity? You may assume that the probe fell in radially from rest at infinity as in the discussion above. (b) Evaluate this time interval for the case of a black hole with a mass $3M_\odot$ as well as the case of a supermassive black hole with a mass of $10^9 M_\odot$.

7.1A (a) To arrive at the proper time interval of going from r^* to 0 : $\Delta\tau = [\tau(r=0) - \tau(r=r^*)] = 2r^*/3c$, we need to use (7.4) for $\tau(r)$:

$$\begin{aligned}\tau(0) &= \tau_0 + \frac{2r^*}{3c}\left(\frac{r_0}{r^*}\right)^{\frac{3}{2}} \\ \tau(r^*) &= \tau_0 - \frac{2r^*}{3c}\left[1 - \left(\frac{r_0}{r^*}\right)^{\frac{3}{2}}\right].\end{aligned} \tag{E.152}$$

(b) For the case of a black hole with a mass $3M_\odot$ we have $\Delta\tau = 2r_\odot^*/c$. With $r_\odot^* = 3$ km the time interval is then $\Delta\tau = 2\times 10^{-5}{\rm s} = 20\mu s$. A supermassive black hole of $10^9 M_\odot$ mass has a Schwarzschild radius $r_{SMBH}^* = 10^9 r_\odot^*$. The infalling time comes out to be $\Delta\tau_{SMBH} \simeq 2\times 10^9\ km/c \simeq 2$ hours.

(7.2) Retarded EF coordinates with past $r = 0$ singularity: Above we obtained the advanced EF coordinates $(\bar{t}, r)$ with lightlike geodesics, (7.14) and (7.15), defining lightcones tilting over smoothly inward towards a future $r = 0$ singularity. Obtain likewise the corresponding retarded EF coordinates $(\tilde{t}, r)$. Find the outgoing and incoming light geodesics that bound lightcones tilting outward away from a past $r = 0$ singularity as in Fig. 7.3(b).

7.2A Here one wants the outgoing light geodesics be represented by 45^o world-lines $cd\tilde{t} = dr$, instead of $cd\bar{t} = -dr$.This suggests that in the $ds^2 = 0$

equation (7.11) one makes the identification of

$$\left(cdt - \frac{dr}{1 - r^*/r}\right) = \left(cd\tilde{t} - dr\right). \tag{E.153}$$

The vanishment of the other parenthesis in (7.11) then leads to light geodesic relations, instead of (7.15), of

$$cd\tilde{t} = -\frac{r + r^*}{r - r^*}dr. \tag{E.154}$$

This equation describes *outgoing* light geodesics in the $r < r^*$ region and *incomi*ng light geodesics in the $r > r^*$ region, as shown in Fig.7.3(b).

(7.3) Change of BH mass is proportional to BH gravity and change of area: Using the definition of BH surface gravity of (7.25) to derive the mass/area relation shown in (7.26).

7.3A The BH area being

$$A^* = 4\pi r^{*2} = \frac{16\pi G_N^2 M^2}{c^4}$$

or

$$dA^* = \frac{32\pi G_N^2 M}{c^4}dM = \frac{8\pi G_N}{\sigma^*}dM$$

where we have plug in the definition of σ^*.

(7.4) From Hawking temperature to the proportionality of BH entropy to area: By a simple integration of $dS = T^{-1}dE$, derive the proportionality of BH entropy and area as shown in (7.33).

7.4A With $E = Mc^2$ and the Hawking temperature of $k_B T = \frac{\hbar c^3}{8\pi G_N M}$, we have

$$\frac{dS}{k_B} = \frac{c^2}{k_B T}dM = \frac{4\pi G_N}{\hbar c}dM^2$$

which leads to

$$\begin{aligned} \frac{S}{k_B} &= \frac{4\pi G_N}{\hbar c}M^2 \\ &= \frac{1}{4}4\pi\left(\frac{2G_N M}{c^2}\right)^2 \frac{c^3}{\hbar G_N} = \frac{A^*}{4l_{Pl}^2}. \end{aligned} \tag{E.155}$$