The solutions manual - Chapter 8
A College Course on Relativity and Cosmology by Ta-Pei Cheng

(8.1) Dimension of critical density: Check the right-hand-side of (8.14) to ensure that it has the correct dimension of [mass]/[volume].

8.1A The Hubble constant has inverse time dimension $[H] = \left[t^{-1}\right]$, and gravitational energy $G_{\mathrm{N}} m^2/r$ has the dimension of $\left[mv^2\right]$, or $[G_{\mathrm{N}}] = \left[l^3/\left(mt^2\right)\right]$, hence

$$\frac{\left[H^2\right]}{[G_{\mathrm{N}}]} = \frac{\left[t^{-2}\right]}{[m^{-1}l^3/t^2]} = \frac{[m]}{[l^3]}. \tag{E.156}$$

(8.2) Friedmann equations and energy conservation: Show that a linear combination of Friedmann (8.42) and (8.43) leads to

$$\frac{d}{dt}(\rho c^2 a^3) = -p\frac{da^3}{dt}, \tag{E.157}$$

which, having the form of the First Law of thermodynamics $dE = -pdV$, is the statement of energy conservation.

8.2A By differentiating (8.42) we have

$$2\dot{a}\ddot{a} = \frac{8\pi G_{\rm N}}{3}\frac{d}{dt}\left(\rho a^2\right) \tag{E.158}$$

and noting that

$$\frac{d}{dt}\left(\rho a^3\right) = a\frac{d}{dt}\left(\rho a^2\right) + \rho a^2\dot{a} \tag{E.159}$$

so that (E.158) becomes

$$2\dot{a}\ddot{a} = 8\pi G_{\rm N}\left[\frac{1}{3a}\frac{d}{dt}\left(\rho a^3\right) - \frac{1}{3}\rho a\dot{a}\right]. \tag{E.160}$$

On the other hand, (8.43) can be rewritten as

$$2\dot{a}\ddot{a} = -8\pi G_{\rm N}\left(\frac{pa\dot{a}}{c^2} + \frac{1}{3}\rho a\dot{a}\right). \tag{E.161}$$

Equating these two equations, we immediately obtain

$$\frac{d}{dt}(\rho c^2 a^3) = -p\frac{da^3}{dt}. \tag{E.162}$$

(8.3) Newtonian interpretation of the second Friedmann equation: Show that the second Friedmann equation (8.43) can be viewed as the $F = ma$ equation in a Newtonian system (i.e., a system with negligible pressure $p = 0$).

8.3A For Newtonian matter (hence the pressure $p = 0$), the gravitational attraction by the whole sphere in Fig. 8.6, being $-G_{\rm N}M/r^2 = \ddot{r}$, or

$$-\frac{4\pi G_{\rm N} a\rho}{3} = \ddot{a} \tag{E.163}$$

which is just (8.43) without the pressure term as appropriate for nonrelativistic matter.

(9.1) Photon temperature boost by e^+e^- annihilation: Since neutrinos and photons were once coupled and in thermal equilibrium, their temperatures were the same: $T'_\nu = T'_\gamma$. The reaction $e^+ + e^- \leftrightarrows \gamma + \gamma$ ceased to proceed from right to left when the photon energy fell below 0.5 MeV.

The disappearance of positrons increased the photons' number and hence their temperature. This temperature boost can be calculated through the entropy conservation condition. Entropy (S) is related to energy (U) as $dS = (1/T)\,dU = (V/T)\,du$, where V and u are respectively the volume and energy density. Given $u \propto g^* T^4$ from (9.3), the key entropy dependences can be identified as $S \propto g^* V T^3$. By comparing the volume and photon temperature change as required by the entropy conservation condition $S'_{e^+} + S'_{e^-} = S_\gamma - S'_\gamma$ in the annihilation reaction and the corresponding volume and neutrino temperature change for the uncoupled neutrinos $S'_\nu = S_\nu$, show that the final photon and neutrino temperatures are related by (9.37): $T_\gamma = (11/4)^{1/3}\, T_\nu$.