The solutions manual - Chapter 9
A College Course on Relativity and Cosmology by Ta-Pei Cheng

(9.1) Photon temperature boost by e^+e^- annihilation: Since neutrinos and photons were once coupled and in thermal equilibrium, their temperatures were the same: $T'_\nu = T'_\gamma$. The reaction $e^+ + e^- \leftrightarrows \gamma + \gamma$ ceased to proceed from right to left when the photon energy fell below 0.5 MeV.

The disappearance of positrons increased the photons' number and hence their temperature. This temperature boost can be calculated through the entropy conservation condition. Entropy (S) is related to energy (U) as $dS = (1/T)\,dU = (V/T)\,du$, where V and u are respectively the volume and energy density. Given $u \propto g^* T^4$ from (9.3), the key entropy dependences can be identified as $S \propto g^* V T^3$. By comparing the volume and photon temperature change as required by the entropy conservation condition $S'_{e^+} + S'_{e^-} = S_\gamma - S'_\gamma$ in the annihilation reaction and the corresponding volume and neutrino temperature change for the uncoupled neutrinos $S'_\nu = S_\nu$, show that the final photon and neutrino temperatures are related by (9.37): $T_\gamma = (11/4)^{1/3}\, T_\nu$.

9.1A The entropy conservation condition for the e^+e^- annihilation reaction may be written as

$$S'_\gamma + S'_{e^+} + S'_{e^-} = S_\gamma. \tag{E.164}$$

Given that $T'_\gamma = T'_{e^+} = T'_{e^-}$, writing out the effective spin degrees g^* of (9.5) this entropy conservation condition becomes

$$[2 + \frac{7}{8}\,(2+2)]V'T'^3_\gamma = 2VT^3_\gamma \tag{E.165}$$

or

$$\frac{V'}{V} = \frac{4}{11}\left(\frac{T_\gamma}{T'_\gamma}\right)^3. \tag{E.166}$$

The volume factor can be replaced by neutrino temperature ratio because of entropy conservation of the uncoupled neutrinos $V'T'^3_\nu = VT^3_\nu$:

$$\left(\frac{T_\nu}{T'_\nu}\right)^3 = \frac{4}{11}\left(\frac{T_\gamma}{T'_\gamma}\right)^3. \tag{E.167}$$

Before the positron disappearance, neutrino and photon temperatures are the same, $T'_\nu = T'_\gamma$, leading to the stated relation (9.37).

(9.2) The radiation and matter equality time: The early universe was radiation-dominated; it then gave way to a matter-dominated system. The radiation-matter equality time t_{RM} is defined to be the cosmic time when the energy densities of radiation and matter were equal:

$$1 = \frac{\rho_R(t_{RM})}{\rho_M(t_{RM})} = \frac{\Omega_R(t_{RM})}{\Omega_M(t_{RM})}. \tag{E.168}$$

Calculate t_{RM} by the steps outlined below:

(a) From the scaling behavior of the radiation and matter densities, relate the scale factor a_{RM} at radiation-matter equality time to the matter-to-radiation density ratio now, $\Omega_R(t_0)/\Omega_M(t_0)$.

(b) This density ratio can be calculated to have the value of $\simeq 3 \times 10^{-4}$ with the following inputs: radiation is composed of photons and neutrinos, so the total radiation energy density $\Omega_R(t_0)$ is related to photon density by way of (9.40); the matter content $\Omega_M(t_0)$ can be deduced from the baryon fraction of matter $\Omega_B(t_0)/\Omega_M(t_0) \simeq 0.05/0.31$ and the photon-to-baryon number ratio n_γ/n_B. The energy of baryonic matter $E_B(t_0)$ can be calculated by adding up the nucleon rest energies, while the photon energy $\bar{E}_\gamma(t_0)$ can be deduced from its value of $0.7\ eV$ at redshift of $z_\gamma \simeq 1100$.

(c) Follow the worked Example of 9.2, using the result for a_{RM} from parts (a) and (b) and a cosmic age $t_0 = 14\ Gyr$ to show that the radiation-matter dominance transition happened approximately 73000 years after the big bang.

9.2A (a) Since radiation density scales as a^{-4} and matter scales as a^{-3}, we have

$$1 = \frac{\rho_R(t_{RM})}{\rho_M(t_{RM})} = \frac{\rho_R(t_0)/a^4(t_{RM})}{\rho_M(t_0)/a^3(t_{RM})}, \qquad \text{or} \qquad a(t_{RM}) = \frac{\Omega_R(t_0)}{\Omega_M(t_0)}. \tag{E.169}$$

(b) We first relate this density ratio through matter's baryon content with the present fraction being $\Omega_B(t_0)/\Omega_M(t_0) = 0.05/0.31$, and relate the radiation density to photon density $\Omega_R(t_0) = 1.68\Omega_\gamma(t_0)$

$$\begin{aligned} a(t_{RM}) &= \frac{\Omega_R(t_0)}{\Omega_M(t_0)} = \frac{1.68\Omega_\gamma(t_0)}{\Omega_B(t_0)}\frac{\Omega_B(t_0)}{\Omega_M(t_0)} \\ &= 1.68 \times \frac{n_\gamma}{n_B}\frac{E_\gamma(t_0)}{E_B(t_0)}\frac{\Omega_B(t_0)}{\Omega_M(t_0)} \\ &= 1.68 \times \frac{10^9}{0.6} \times \frac{0.7eV/1100}{939 \times 10^6 eV} \times \frac{0.05}{0.31} \\ &\simeq 3 \times 10^{-4}. \end{aligned} \tag{E.170}$$

It then follows that $\Omega_R(t_0) = 3 \times 10^{-4}\Omega_M(t_0)$. Clearly we are in a matter dominated epoch with $\Omega_M(t_0) = 0.31$ and $\Omega_R(t_0) \simeq 1 \times 10^{-4}$.

(c) Assuming that the universe has been matter dominated for $t > t_{RM}$, the cosmic time dependence of the scale factor should be $a \propto t^{2/3}$.

$$\frac{a_0}{a_{RM}} = \left(\frac{t_0}{t_{RM}}\right)^{2/3} \tag{E.171}$$

or

$$\begin{aligned} t_{RM} &= (a_{RM})^{3/2} t_0 \\ &= \left(3 \times 10^{-4}\right)^{3/2} \times 1.4 \times 10^{10}\ yr = 7.3 \times 10^4\ yr. \end{aligned} \tag{E.172}$$

(9.3) Temperature dipole anisotropy as Doppler effect: By converting temperature variation to that of light frequency $T \sim 1/a \sim 1/lengh$, show that the Doppler effect (3.52) implies that an observer moving with a nonrelativistic velocity $\mathbf{v}$ through an isotropic CMB would see a temperature dipole anisotropy of $\delta T/T = \frac{v}{c}\cos\theta$, where the angle θ is measured from the direction of the motion.

9.3A Recall that temperature scales as a^{-1}, that is, as inverse wavelength, or as frequency:

$$\frac{\delta T}{T} = \frac{\delta\omega}{\omega}. \tag{E.173}$$

But the nonrelativistic Doppler effect (the small β limit of (3.52) reads

$$\omega' = \left(1 - \frac{v}{c}\cos\theta\right)\omega \tag{E.174}$$

or $(\delta\omega/\omega) = (v/c)\cos\theta$.