The solutions manual - Chapter 10
A College Course on Relativity and Cosmology by Ta-Pei Cheng

(10.1) Estimate $|1-\Omega(t_{\rm bbn})|$ from $|1-\Omega(t_{\rm RM})|$: Given that the universe was radiation-dominated from $t_{\rm bbn}$ to $t_{\rm RM}$, estimate the magnitude of $|1-\Omega(t_{\rm bbn})|$ from $|1-\Omega(t_{\rm RM})|$,as in (10.2). The estimate will depend on knowing that the scale factor evolves as the inverse temperature $a_{\rm RM}/a_{\rm bbn} = T_{\rm bbn}/T_{\rm RM} = O\left(10^5\right)$.

10.1A The time evolution for cosmic time (including $t_{\rm bbn}$) prior to $t_{\rm RM}$ was radiation dominated: $a(t) \sim t^{1/2}$ or $\dot{a} \sim t^{-1/2} \sim a^{-1}$, we have from (10.2),

$$\frac{1-\Omega(t_{\rm bbn})}{1-\Omega(t_{\rm RM})} = \left[\frac{\dot{a}(t_{\rm bbn})}{\dot{a}(t_{\rm RM})}\right]^{-2} = \left[\frac{a(t_{\rm bbn})}{a(t_{\rm RM})}\right]^{2}. \qquad \text{(E.175)}$$

Combining the results of $a_{\rm RM}/a_{\rm bbn} = T_{\rm bbn}/T_{\rm RM} = O\left(10^5\right)$ and $|1-\Omega(t_{\rm RM})| = O\left(10^{-4}\right)$ from (10.2), we have

$$\begin{aligned} |1-\Omega(t_{\rm bbn})| &= |1-\Omega(t_{\rm RM})| \left[\frac{T_{\rm RM}}{T_{\rm bbn}}\right]^2 \qquad \text{(E.176)} \\ &= O\left(10^{-4}\right)\left[O\left(10^{-5}\right)\right]^2 = O\left(10^{-14}\right). \end{aligned}$$

(10.2) Redshift dependence of Hubble constant via the energy densities: (a) Insert the present Friedmann equation (8.48) into the general equation (8.42) and then use (8.14) for the critical density to express the epoch-dependent Hubble parameter as $H(a) = H_0\left[\rho(a)/\rho_{c,0} + (1-\Omega_0)/a^2\right]^{1/2}$ (b) Put in the scale dependence of the densities in $\rho(a) = \rho_M(a) + \rho_\Lambda$ (neglect the radiation density, which is negligible long after t_{RM}) to find

$$H(a) = H_0\left(\frac{\Omega_{\rm M,0}}{a^3} + \Omega_\Lambda + \frac{1-\Omega_0}{a^2}\right)^{1/2}, \qquad \text{(E.177)}$$

or in terms of the redshift in (8.35),

$$H(z) = H_0[\Omega_{\rm M,0}(1+z)^3 + \Omega_\Lambda + (1-\Omega_{\rm M,0}-\Omega_\Lambda)(1+z)^2]^{1/2}. \qquad \text{(E.178)}$$

10.2A (a) First we replace the curvature parameter k term in (8.42) by the expression shown in (8.47),

$$[H(a)]^2 = \frac{\dot{a}^2(t)}{a^2(t)} = \frac{8\pi G_{\mathrm{N}}}{3}\rho + \frac{\dot{a}^2(t_0)}{a^2(t)}(1-\Omega_0). \tag{E.179}$$

Pulling out the $\dot{a}^2(t_0) = H_0^2$ and identify the combination$3H_0^2/(8\pi G_{\mathrm{N}})$ as the critical density now $\rho_{\mathrm{c},0}$, we have

$$[H(a)]^2 = H_0^2\left(\frac{\rho(a)}{\rho_{\mathrm{c},0}} + \frac{1-\Omega_0}{a^2(t)}\right). \tag{E.180}$$

(b) With $\rho(a)$ expressed in terms of the various components

$$\frac{\rho(a)}{\rho_{\mathrm{c},0}} = \Omega(a) = \frac{\Omega_{\mathrm{R},0}}{a^4} + \frac{\Omega_{\mathrm{M},0}}{a^3} + \Omega_{\Lambda,0}, \tag{E.181}$$

and dropping the negligible radiation density $\Omega_{\mathrm{R},0}$, (E.180) becomes, after dropping the negligible radiation density $\Omega_{\mathrm{R},0}$,

$$H(a) = H_0\left(\frac{\Omega_{\mathrm{M},0}}{a^3} + \Omega_\Lambda + \frac{1-\Omega_0}{a^2}\right)^{1/2}, \tag{E.182}$$

which can in turn be expressed in terms of the redshift,

$$H(z) = H_0[\Omega_{\mathrm{M},0}(1+z)^3 + \Omega_\Lambda + (1-\Omega_{\mathrm{M},0}-\Omega_\Lambda)(1+z)^2]^{1/2}. \tag{E.183}$$

(10.3) A check on the cosmic age formula: For a matter-dominated flat universe without a cosmological constant ($\Omega_0 = \Omega_{M,0} = 1$ with $\Omega_{\Lambda,0} = 0$), check that the cosmic age formula of (10.26) has the correct limit of $t_0 = \frac{2}{3}t_H$ in agreement with the result obtained in (8.71).

10.3A Plug in the condition of $\Omega_{M,0} = 1$ into (10.26) and take the $\Omega_\Lambda \to 0$ limit of $\left[\ln\left(\sqrt{\Omega_\Lambda} + \sqrt{1+\Omega_\Lambda}\right)\right] \to \sqrt{\Omega_\Lambda}$, we obtain $t_0 = \frac{2}{3}t_{\mathrm{H}}$.

(10.4) Estimate of matter and dark energy equality time Closely related to the deceleration/acceleration transition time t_{tr} is the epoch $t_{\mathrm{M}\Lambda}$ when the matter and dark energy components are equal $\Omega_{\mathrm{M}}(t_{\mathrm{M}\Lambda}) = \Omega_\Lambda(t_{\mathrm{M}\Lambda})$. Show that the estimated $t_{\mathrm{M}\Lambda}$ is comparable to, and as expected somewhat greater than, $t_{\mathrm{tr}} \simeq 7.6$ Gyr obtained above.

10.4A The matter and dark energy equality time being defined by $\Omega_{\mathrm{M}}(t_{\mathrm{M}\Lambda}) = \Omega_\Lambda(t_{\mathrm{M}\Lambda})$, from their respective scaling property, we can then relate the matter and dark energy densities now to the scale factor $a(t_{\mathrm{M}\Lambda})$

$$[a(t_{\mathrm{M}\Lambda})]^3 = \frac{\Omega_{\mathrm{M},0}}{\Omega_\Lambda} \tag{E.184}$$

Plugging this into (10.25)

$$\frac{t_{\mathrm{M}\Lambda}}{t_{\mathrm{H}}} = \frac{2}{3\sqrt{\Omega_\Lambda}} \ln\left(1 + \sqrt{2}\right) = 0.71. \tag{E.185}$$

Thus the matter and dark energy equality time $t_{\mathrm{M}\Lambda} \simeq 10.2$ Gyr.