Supplementary note on Tensors: a practical lesson

Tensors are mathematical objects that have definite transformation properties under some coordinate transformations (for example, rotations, Lorentz transformations, etc.) Examples of tensors are vectors (tensor of rank-1), scalars (tenors of rank-0), or tensors of higher ranks – e.g. rank-2 tensors: electromagnetic field tensor (Sec 3.2.3), energy momentum tensor (Sec 3.2.4). The reason we are interested in tensor equations because they are automatically coordinate symmetric (relativistic), See *e.g*. Eq.(1.8).

In this course we shall deal mainly with tensor components with contravariant components having upper indices and covariant components having lower indices, Eq.(3.12). They transform oppositely Eq.(3.27) and video lecture #3c. Examples of tensors:

- Rank-1 (vectors): $A'^{\alpha} = [R]^{\alpha}_{\beta} A^{\beta}, \quad A'_{\alpha} = [R^{-1}]^{\beta}_{\alpha} A_{\beta}$
    - $[R]$ and $[R^{-1}]$ are respectively the transformation matrix and its inverse .
    - α's are free indices— they must match in each term of a tensor equation.
    - Each of the above equations has a pair repeated indices (β's), meaning they are summed over. Namely, here we adopt Einstein summation convention of omitting the display of the summation sign $\sum_{\beta}$
- Rank-2 tensors:
  $$A'^{\alpha\gamma} = [R]^{\alpha}_{\beta}[R]^{\gamma}_{\delta} A^{\beta\delta}, \quad A'_{\alpha\gamma} = [R^{-1}]^{\beta}_{\alpha}[R^{-1}]^{\delta}_{\gamma} A_{\beta\delta}, \quad A'^{\alpha}_{\gamma} = [R]^{\alpha}_{\beta}[R^{-1}]^{\delta}_{\gamma} A'^{\beta}_{\delta}$$
    - α's and γ's are free indices; each term must be of the same rank.
    - The pairs of β's and δ's are repeated indices, meaning they are summed over. Namely, there are two summations in the above equations, but we omit the display of the summation signs $\sum_{\beta}$ and $\sum_{\delta}$
    - For each upper index of a tensor, there is a $[R]$ transformation, and each lower a $[R^{-1}]$.
- Rank-3 tensors:
  $$A'^{\alpha\gamma\rho} = [R]^{\alpha}_{\beta}[R]^{\gamma}_{\delta}[R]^{\rho}_{\sigma} A^{\beta\delta\sigma}, \quad A'^{\alpha}_{\gamma\rho} = [R]^{\alpha}_{\beta}[R^{-1}]^{\delta}_{\gamma}[R^{-1}]^{\sigma}_{\rho} A'^{\beta}_{\delta\sigma}, \quad etc.$$

In the following I will comment on a couple of homework assignments related to tensors

[I] Ex 3.2 on p.38 (the quotient theorem) is one of the assigned homework problems. Here I will work out a slightly simpler version and you should be able complete the exercise by following the same steps.

■ Given a tensor equation $A^{\alpha} = C^{\alpha\beta} B_{\beta}$. If we know that A^{α} and B_{β} are tensors, prove that $C^{\alpha\beta}$ must also be a tensor. Namely, demonstrate that $C'^{\alpha\beta} = [R]^{\alpha}_{\gamma}[R]^{\beta}_{\varepsilon} C^{\gamma\varepsilon}$.

The proof goes like this: $A^{\alpha} = C^{\alpha\beta} B_{\beta}$ being a tensor equation, it then holds not only in O-frame, but must also in the *O'*-frame: $A'^{\alpha} = C'^{\alpha\beta} B'_{\beta}$. Since A^{α} and B_{β} are tensors, we have $A'^{\alpha} = [R]^{\alpha}_{\gamma} A^{\gamma}$ and $B'_{\beta} = [R^{-1}]^{\delta}_{\beta} B_{\delta}$. Sub them into the primed relation, we then have

$$[R]^{\alpha}_{\gamma} A^{\gamma} = C'^{\alpha\beta} [R^{-1}]^{\delta}_{\beta} B_{\delta}$$

.

Now, sub in on the LHS the unprimed relation $A^\gamma = C^{\gamma\varepsilon}B_\epsilon$, we get

$$[R]^\alpha_\gamma C^{\gamma\varepsilon}B_\epsilon = C'^{\alpha\beta}[R^{-1}]^\delta_\beta B_\delta.$$

We can "move" the $[R^{-1}]^\delta_\beta$ factor from RHS to the LHS by multiplying both sides with $[R]^\beta_\varepsilon$ and noting $[R][R^{-1}] = [I]$ so that

$$[R]^\beta_\varepsilon[R]^\alpha_\gamma C^{\gamma\varepsilon}B_\epsilon = C'^{\alpha\beta}\{[R]^\beta_\varepsilon[R^{-1}]^\delta_\beta\}B_\delta = C'^{\alpha\beta}\delta^\delta_\varepsilon B_\delta = C'^{\alpha\beta}B_\varepsilon,$$

where we have replaced the product factor {... } in the 2nd term by a Kronecker delta $\delta^\delta_\varepsilon$. Finally, cancelling the common factor B_ϵ in the first with the one in last term, we have $[R]^\beta_\varepsilon[R]^\alpha_\gamma C^{\gamma\varepsilon} = C'^{\alpha\beta}$, proving that $C^{\alpha\beta}$ is a tensor.

One of the practical lessons I want to emphasize here is that $[R]^\beta_\varepsilon$'s are components of a matrix. While matrices are non-commutative, their components are just ordinary numbers that we can freely move them around. The same property holds for tensor components too.

[II] Now on to Ex 3.9 – Lorentz transformation of EM fields from the covariant formalism. You are asked to show that result of Eq.(2.40) is contained in Eq.(3.69). First, the indices $\mu, \nu, ..$ run over the range of 0, 1, 2, 3 with $1 = x, 2 = y, 3 = z$. From Eq.(3.63), or (3.64), noting $F_{01} = -E_x$, we then concentrate on Eq. (3.69), with the free indices μ and ν set at $\mu = 0, \nu = 1$ so that $F'_{01} = [L^{-1}]^\lambda_0[L^{-1}]^\rho_1 F_{\lambda\rho}$. This involves a double sum over the repeated indices of λ as well as ρ (with the summation signs $\sum_\lambda$ and $\sum_\rho$ omitted). In principle there are 16 terms on the RHS; however many of these terms vanish as $[L^{-1}]^2_0 = [L^{-1}]^3_1 = F_{00} = F_{11} = 0$, all we have $F'_{01} = [L^{-1}]^0_0[L^{-1}]^1_1 F_{01} + [L^{-1}]^1_0[L^{-1}]^0_1 F_{10}$.

Lorentz transformation being $[L^{-1}] = \begin{pmatrix} \gamma & \beta\gamma & & \\ \beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$, with $F_{01} = -E_1 = -E_x$, we obtain

$-E'_x = \gamma^2(-E_x) + \beta^2\gamma^2 E_x = -E_x$, after $\gamma^2(1-\beta^2) = 1$. This is one of the Eq. (2.40) result.

This is part (a) of the homework. For part (b), you are asked to convert Eq.(3.69) into matrix multiplication. The key point keep in mind is that in a matrix component representation of $[A]^\alpha_\beta$, or $F_{\alpha\beta}$, the α is the column index while β is the row index. The product of matrices

involves the multiplication of row element with the column element: hence the pair-wise contraction of column index with the correspondent row index. We need to place the contracted indices next to each other: $F'_{\mu\nu} = [L^{-1}]^\lambda_\mu[L^{-1}]^\rho_\nu F_{\lambda\rho} = \{[L^{-1}]^\lambda_\mu F_{\lambda\rho}\}[L^{-1}]^\rho_\nu$. Namely, the standard matric multiplication of $[L^{-1}]$[F] with $[L^{-1}]$ on the left, then multiply $[L^{-1}]$ from right. However its column and row indices are at the wrong place. Thus we need to take the "transpose" of the matrix. But in our case $[L^{-1}]$ is a symmetric matrix, *i.e.*, transpose operation has no effect. The final result is the matrix multiplication $[F'] = [L^{-1}][F][L^{-1}]$ as shown at the last line on p.271.